A TEXT-BOOK

ON

ROOFS AND BRIDGES.

PART IV.

HIGHER STRUCTURES.

BY

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PREFACE.

Parts I, II, and III of this Text-Book are devoted almost entirely to framed structures having two supports whose reactions are vertical. These probably constitute more than ninety per cent of all roof and bridge trusses, and hence should claim the greater part of the time of the student. In this volume are discussed those structures which have more than two supports, as continuous, draw and cantilever bridges, or which have two supports whose reactions are not vertical, as the suspension and arched systems.

The investigations here given are mainly those of the theory of stresses and their determination by analytic or graphic methods. The continuous girder is treated with less fullness than usual, but sufficiently to develope the necessary formulas for swing bridges. Partially continuous swing bridges are discussed in detail and an exact method is given for finding the true reactions and stresses. Cantilever and suspension structures are treated more fully than is usual in American books, and critical analyses regarding economic proportions and the limitations of the theory are presented. The discussion of the three-hinged arch is also given in detail, the actual maximum and minimum stresses being computed for several cases. For arches with two hinges and with no hinges the reactions are determined analytically while the stresses are derived by simple graphic constructions. In finding the conditions of loading and the stresses the effort has been made to develope those analytic and graphic methods which seem simplest in theory and most expeditious in practical use. The attempt has been made throughout to present the subject clearly and concisely, to incite interest by giving historical information, and to exemplify the theory by illustrative examples and problems from the best engineering practice.

The word 'kip' is used in this volume to denote one thousand pounds. As one thousand grams is a kilogram which is usually abbreviated into kilo, so one thousand pounds may be called a kilopound and be abbreviated into kip. Since stresses are now generally computed in thousands of pounds instead of in tons or pounds, a name for the new unit is advantageous, and after using it for several years in the classroom it is thought best to formally suggest its general adoption.

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CONTENTS.

CHAPTER I.

CONTINUOUS BRIDGES.

CHAPTER II.

DRAW BRIDGES.

Chapter III.

CANTILEVER BRIDGES.

CHAPTER IV.

SUSPENSION BRIDGES.

CHAPTER V.

THREE-HINGED ARCHES.

Chapter VI.

TWO-HINGED ARCHES.

CHAPTER VII.

ARCHES WITHOUT HINGES.

HIGHER STRUCTURES.

CHAPTER I.

CONTINUOUS BRIDGES.

ART. 1. INTRODUCTION.

Probably over ninety per cent of all roof and bridge structures are formed by the use of the simple beams or trusses which are analysed and discussed in Parts I, II, and III of this text-book. Other kinds of trusses may be grouped together under the name of Higher Structures, a term which implies that they are more complex in theory and design, and that they are built only for long spans or under special circumstances to which simple trusses do not economically apply.

A simple beam or truss rests upon two supports, and exerts only vertical pressures upon those supports. A continuous, draw, or cantilever truss rests upon more than two supports, but exerts only vertical pressures upon them. A suspension or arched structure exerts horizontal as well as vertical pressures upon its supports.

A beam or truss which rests upon more than two supports, and has no joints or hinges to prevent the full transmission of shears and moments from one span to another, is said to be continuous, or fully continuous. Such is the case with a solid beam laid upon several supports, like a floor girder or a railroad rail; such is the case with a trussed structure which has its chords and webbing extended without interruption over its entire length. Many swing draw bridges have such continuous trusses.

For a simple beam or truss the reactions due to a given load are at once found by the principles of statics, and the stresses due to the load are then computed. For a continuous beam or truss, however, the reactions due to a given load cannot be found by pure statics, as they are greater in number than the statical conditions of equilibrium. To find the reactions for continuous beams, an additional condition is to be introduced from the theory of elasticity by means of the equation of the elastic curve. After the reactions are computed, the shears and moments due to given loads are readily determined either analytically or graphically.

Continuous bridges were extensively built in Europe between the years 1850 and 1870, the number of spans in a structure being usually three, four, or five, and the length of span ranging from 125 to 250 feet. They have not been used in the United States except for swing draw bridges, of which many have been erected over navigable rivers. As a rule, when several spans are required to cross a river several independent simple trusses are preferable to one continuous structure, as in the latter case great changes in stresses may be caused by very slight variations in the level of the piers. The general theory of continuity is, however, of great importance on account of the extensive use of continuous solid beams in building construction, as well as the continuous draw bridges and other partially continuous structures used in cantilever and suspension systems.

ART. 2. VERTICAL SHEARS AND BENDING MOMENTS.

If a beam or truss be cut by a vertical plane, all the internal forces or stresses in the section may be resolved into horizontal and vertical components. The sum of the horizontal components is zero, since all the external forces or loads are vertical. The sum of the vertical components is the resistance to shearing in that vertical plane, and is called the internal shear, or simply the shear. Further, the sum of the moments of all these stresses about any point in the plane is called the resisting moment. From the internal shear and the resisting moment the stresses themselves are determined.

Since the girder is in equilibrium, the internal shear and resisting moment are balanced by the vertical shear and the bending moment of the external forces acting on either side of the section. The algebraic sum of all the external forces on the left of the section is called the vertical shear V, and the algebraic sum of the moments of all the external forces on the left of the section with respect to a point in that section is called the bending moment M. Thus the internal shear and moment are equal respectively to V and M.

From the definitions just stated the values of V and M are readily written for any section of a continuous girder loaded in any manner, provided that the reactions of the supports be known. For instance, let the upper diagram in Fig. 1, represent a beam of three equal spans loaded with w per linear unit. Let l be the length of span, then the total load is $3\,wl$, and by methods which will be given later the reactions may be found to be $R_1 = R_4 = 0.4\,wl$, and $R_2 = R_3 = 1.1\,wl$. Now let a section be taken in the middle span at a distance x from the second support, then for this section,

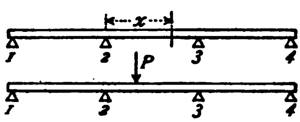

Fig. 1.

$$V = R_1 + R_2 - w(l + x) = 0.5\,wl - wx,$$

$$M = R_1(l + x) + R_2x - w(l + x)\frac{l + x}{2}$$

$$= -0.1\,wl^2 + 0.5\,wlx - 0.5\,wx^2,$$

are expressions for the vertical shear and the bending moment.

Again, in the lower diagram of Fig. 1, let a single load P be on the middle span at a distance $\frac{1}{3}l$ from the support *2*; for this case the reactions are $R_1 = -\frac{32}{405}P$, $R_2 = +\frac{312}{405}P$, $R_3 = +\frac{147}{405}P$, and $R_4 = -\frac{22}{405}P$. Now for a section in the middle span distant x from the support *2* and beyond the load, the shear is

$$V = R_1 + R_2 - P = -\frac{125}{405}P,$$

and the moment is expressed by

$$M = R_1(l + x) + R_2x - P(x - \tfrac{1}{3}l) = +\frac{103}{405}Pl - \frac{125}{405}Px,$$

in which x may have any value between $\frac{1}{3}l$ and l.

For a continuous truss with parallel chords, as is generally the case, where the webbing is such that only one web member is cut by the section, the stress in that member is $V \sec\theta$, where θ is the angle which the member makes with the vertical (Part I, Art. 26). Further, if the section pass through a center of moments for a chord member, the stress for that chord member is M/d, where d is the lever arm of the chord member, or the depth of truss when the chords are parallel. Thus when V and M have been computed for given loads at all sections, the stresses due to those loads are readily found for all truss members.

For a continuous solid beam the values of V and M serve to determine the unit stresses of shearing in the cross-section and of tension and compression on the sides of the beam, by the methods established in Mechanics of Materials, Art. 21.

Problem 1. For the two cases in Fig. 1 derive the values of V and M for a section at the center of the first span, and also for a section in the middle span at a distance of $\frac{1}{4}l$ from the support *2*.

ART. 3. REACTIONS OF SUPPORTS.

The reactions of the supports of a continuous beam are to be determined by first finding, by the method of Art. 4, the bending moments at the supports, and then stating equations of moments for sections at those supports. For example, let

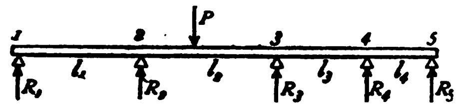

Fig. 2.

there be four spans, as in Fig. 2, where $l_1 = 80$ feet, $l_2 = 100$ feet, $l_3 = 50$ feet, $l_4 = 40$ feet, and let a single load of 1000 pounds be on the second span at a distance of 40 feet from the support *2*. Let the bending moments at the supports be $M_1 = M_4 = 0$ as the ends are not restrained, $M_2 = -8200$, $M_3 = -8856$, $M_4 = +2464$ pound-feet. It is required to compute the reactions.

Taking *2* as a center of moments, the bending moment is $R_1 l_1$ or $80 R_1$, whence $80 R_1 = -8200$ and hence $R_1 = -102.5$ pounds. Again taking *3* as a center of moments,

$$R_1 \times 180 + R_2 \times 100 - P \times 60 = -8856,$$

whence $R_2 = +695.9$ pounds. For the next equation the center is at *4* and there is found $R_3 = +632.9$. To find R_4 and R_5 it is more convenient to use the forces on the right of the section; thus with center at *4* the equation is $R_5 l_5 = +2464$, whence $R_5 = +61.6$. Lastly, with center at *3* there is found $R_4 = -288.0$. The sum of these five reactions equals 1000 pounds, as should be the case.

As a second example, let there be four equal spans uniformly loaded and let the moments at the supports be given as

$$M_1 = M_5 = 0; \quad M_2 = M_4 = -\tfrac{3}{28} wl^2; \quad M_3 = -\tfrac{2}{28} wl^2.$$

To find the reactions, the centers of moments are taken at the supports *2*, *3*, and *4* successively. Thus,

$$R_1 l - \tfrac{1}{2} wl^2 = -\tfrac{3}{28} wl^2,$$

from which $R_1 = +\frac{11}{28} wl^2$; also $R_2 = +\frac{32}{28} wl$, and $R_3 = +\frac{26}{28} wl$. Here, from the symmetry of the spans and load, it is plain that $R_5 = R_1$, and that $R_4 = R_2$.

Prob. 2. A continuous beam has three spans, each 6 feet long, and a load of 1200 pounds at the middle of the first span. The moments M_2 and M_3 are -720 and $+180$ pound-feet respectively. Show that the four reactions are $+480$, $+870$, -180, and $+30$ pounds.

ART. 4. THE THEOREM OF THREE MOMENTS.

The moments at the supports of a continuous girder are deduced by help of the theorem of three moments, a demonstration of which will now be given.

Let l_2 and l_3, in Fig. 3, represent two consecutive spans of a continuous beam, having the loads P_2 and P_3 at the distances kl_2 and kl_3 from the supports *2* and *3*; here k is any fraction less than unity, and not necessarily the same for the two loads.

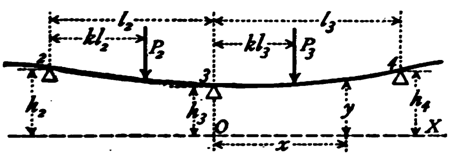

Fig. 3.

Let M be the bending moment for a point of the elastic curve in the span l_3, whose coordinates are x and y. As shown in Mechanics of Materials, Art. 33, the general equation of the elastic curve is

$$\frac{d^2y}{dx^2} = \frac{M}{EI}, \tag{1}$$

where I is the moment of inertia of the cross-section of the beam, and E is the coefficient of elasticity of the material. Now, let V_3 be the resultant of all the vertical forces on the left of P_3, and let v be the distance of its point of application to the left of *3*. The bending moment for the given section is then $M = V_3(v + x) - P_3(x - kl_3)$; or, since $V_3 v$ is M_3, the moment at *3*,

$$M = M_3 + V_3 x - P_3(x - kl_3). \tag{2}$$

If, in this, x be made l_3, the value of M is M_4, and thus

$$V_3 = \frac{M_4 - M_3}{l_3} + P(1 - k). \qquad (3)$$

This is the shear at the right of the support *3*, since V_3 is the resultant of all the vertical forces on the left of the given load P_3.

In the span l_3, the elastic curve on the right of the load will have a different equation from that on the left of the load, as the moment on the right is given by (2), while the moment on the left is simply $M_3 + V_3x$. Inserting this in (1), and integrating twice, the following are found for the elastic curve on the left of the load:

$$EI\frac{dy}{dx} = M_3x + \tfrac{1}{2}V_3x^2 + C, \qquad (4)$$

$$EIy = \tfrac{1}{2}M_3x^2 + \tfrac{1}{6}V_3x^3 + Cx + C_1; \qquad (5)$$

and, similarly, for the elastic curve on the right of the load,

$$EI\frac{dy}{dx} = M_3x + \tfrac{1}{2}V_3x^2 - \tfrac{1}{2}P_3x^2 + Pkl_3x + C'; \qquad (4)'$$

$$EIy = \tfrac{1}{2}M_3x + \tfrac{1}{6}V_3x^3 - \tfrac{1}{6}P_3x^3 + \tfrac{1}{2}Pkl_3x^2 + C'x + C_1'. \qquad (5)'$$

To determine the constants of integration, there are four conditions: first, when $x = 0$ in (5), then $y = h_3$; second, when $x = l$ in (5)′, then $y = h_4$; third, when $x = kl$, the values of dy/dx are the same in (4) and (4)′; and, fourth, when $x = kl$, the values of y are the same in (5) and (5)′.

The constants being found and the value of V_3 inserted from (3) the inclinations of the elastic curve at *3* and *4* are determined by making $x = 0$ in (4) and $x = l$ in (4)′; thus if t_3 and t_4 be these inclinations,

$$t_3 = \frac{h_4 - h_3}{l_3} - \frac{2M_3l_3 + M_4l_3 + P_3l_3^2(2k - 3k^2 + k^3)}{6EI}, \qquad (6)$$

$$t_4 = \frac{h_4 - h_3}{l_3} + \frac{M_3l_3 + 2M_4l_3 + P_3l_3^2(k - k^3)}{6EI}. \qquad (6)'$$

Also, diminishing each subscript in (6)′ by unity,

$$t_3 = \frac{h_3 - h_2}{l_2} + \frac{M_2 l_2 + 2\,M_3 l_2 + P_2 l_2^2(k - k^3)}{6\,EI}, \qquad (7)$$

is the inclination at *3* in terms of the load on the span l_2.

As the two values of t_3 given by (6) and (7) must be equal, since the curve is continuous over the support, there results

$$M_2 l_2 + 2\,M_3(l_2 + l_3) + M_4 l_3 = -P_2 l_2^2(k - k^3)$$
$$- P_3 l_3^2(2k - 3k^2 + k^3) - 6\,EI\left(\frac{h_3 - h_2}{l_2} + \frac{h_3 - h_4}{l_3}\right), \qquad (8)$$

which is the important theorem of three moments applicable to any two consecutive spans of a continuous girder.

If all the supports be on the same level, as is usually the case, the term containing EI reduces to zero. If there be several loads on the spans the sign of summation is to be written before the terms including the loads; thus,

$$M_2 l_2 + 2\,M_3(l_2 + l_3) + M_4 l_3 = -\Sigma P_2 l_2^2(k - k^3)$$
$$- \Sigma P_3 l_3^2(2k - 3k^2 + k^3), \qquad (9)$$

is the theorem for concentrated loads and level supports.

If the two spans be loaded uniformly with w_2 and w_3 per linear unit, the signs of summation are to be replaced by those of integration between the limits 0 and l_2, and 0 and l_3; also P_2 is to be replaced by $w_2 d(kl_2)$, and P_3 by $w_3 d(kl_3)$. Then (9) reduces to

$$M_2 l_2 + 2\,M_3(l_2 + l_3) + M_4 l_3 = -\tfrac{1}{4}\,w_2 l_2^3 - \tfrac{1}{4}\,w_3 l_3^3, \qquad (10)$$

which is the theorem of three moments for uniform loads and level supports, as first announced by CLAPEYRON.

By means of the theorem of three moments an equation may be written for each support of a continuous girder, except those at the ends where the moments are zero. There will be as many equations as there are unknown moments, and from these the values of M_2, M_3, etc., are derived. For instance, considering the four spans of Fig. 2 where a single load is in the span l_2, the theorem for the support *2* is written by diminishing each subscript in (9) by unity, and making $M_1 = 0$ and $P_1 = 0$; thus

$$2M_2(l_1 + l_2) + M_3 l_2 = -Pl_2^2(2k - 3k^2 + k^3).$$

Again for the support *3* the theorem is, since $P_3 = 0$,

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -Pl_2^2(k - k^3),$$

and for the support *4* it is, as there are no loads on l_3 and l_4,

$$M_3 l_3 + 2M_4(l_3 + l_4) = 0.$$

From these three equations the values of M_2, M_3, M_4 are obtained by solution for any given lengths of span. The simplest case is when the spans are all equal; here the solution gives

$$M_2 = -\frac{Pl}{56}(26k - 45k^2 + 19k^3), \quad M_3 = -\frac{Pl}{14}(2k + 3k^2 - 5k^3),$$
$$M_4 = -\tfrac{1}{4}M_3,$$

which are the bending moments at the supports caused by the load P in the second span. If the load is at the middle of the span, the value of k is 0.5, and by giving different values to k the load may have all required positions in the span.

Prob. 3. A continuous beam of four equal spans has a load P on the first span. Show that the moment $M_4 = -\frac{Pl}{56}(k - k^3)$, and that the reaction $R_1 = \frac{P}{56}(56 - 71k + 15k^3)$.

ART. 5. REACTIONS FOR TWO EQUAL SPANS.

By means of the theorem of three moments in Art. 4, the moment M_2 at the middle support of two continuous spans is at once found; then by Art. 3 the reactions of the supports are derived; and lastly, by Art. 2, the shears and moments for every section throughout the beam may be ascertained. All supports are taken on the same level.

CASE I, A Single Load. — Let the load P be in the first span at a distance kl from the left support, l being the length of each span, and k any fraction less than unity. Then since $M_1 = 0$, $M_3 = 0$, and $P_3 = 0$, the theorem in (9) of Art. 4 gives

$$M_2 = -\tfrac{1}{4} Pl(k - k^3),$$

and by the method of Art. 3 the reactions are

$$R_1 = + P(1 - k) - \tfrac{1}{4} P(k - k^3),$$
$$R_2 = + Pk + \tfrac{1}{2} P(k - k^3),$$
$$R_3 = - \tfrac{1}{4} P(k - k^3),$$

the sum of which is equal to P. If the first span were a simple beam the reactions would be $R_1 = P(1 - k)$ and $R_2 = Pk$, but here it is seen that R_1 is less and that R_2 is greater than for the simple beam. The shear and moment diagrams for this case are shown in Fig. 4, the shear being zero and the moment being a maximum under the load. An inflection point, where the moment is zero, lies between the load and the middle support. If x_i be the distance of the inflection point from the support 1, the equation of moments with respect to this point is

$$R_1 x_i - P(x_i - kl) = 0, \quad \text{whence} \quad x_i = \frac{4l}{5 - k^2}.$$

Since the value of k lies between 0 and 1, it is seen that the value of x_i lies between $\tfrac{4}{5} l$ and l, that is, the inflection point is always located on the last fifth of the span.

CASE II, A Uniform Load on One Span. — Let w be the load per linear unit over the first span, the second span being unloaded. The theorem of three moments in (10) of Art. 4 gives $M_2 = - \tfrac{3}{16} wl^2$, and then

$$R_1 = +\tfrac{7}{16}\,wl, \qquad R_2 = +\tfrac{10}{16}\,wl, \qquad R_3 = -\tfrac{1}{16}\,wl.$$

Here zero shear and maximum moment occur at $x = \frac{7}{16}\,l$, while the inflection point is at $\frac{7}{8}\,l$. When x is less than $\frac{7}{8}\,l$, the upper

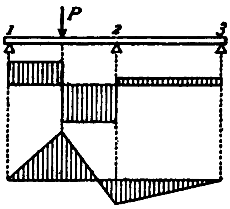

Fig. 4.

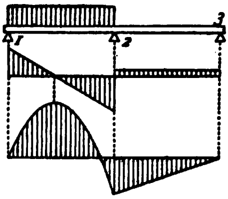

Fig. 5.

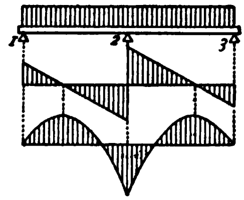

Fig. 6.

part of the beam is in compression and the lower in tension; when x is greater than $\frac{7}{8}\,l$, and also in the second span, the upper part is in tension and the lower in compression.

CASE III, Uniform Load on Both Spans.—For this case it is easy to find, by the same method as before,

$$M_2 = -\tfrac{1}{8}\,wl^2, \qquad R_1 = R_3 = +\tfrac{3}{8}\,wl, \qquad R_2 = +\tfrac{10}{8}\,wl.$$

Zero shear and maximum moment occur when $R - wx = 0$, that is, for $x = \frac{3}{8}\,l$; while the inflection point occurs when $Rx - \frac{1}{2}wx^2 = 0$, that is, for $x_i = \frac{3}{4}\,l$. The distribution of shears and moments throughout the beam is shown in Fig. 6.

Prob. 4. If a load P be on the second span at a distance kl to the left of the support *3*, show that the reactions are

$$R_1 = -\tfrac{1}{4}\,P(k - k^3), \qquad R_2 = +\,Pk + \tfrac{1}{2}\,P(k - k^3),$$
$$R_3 = P(1 - k) - \tfrac{1}{4}\,P(k - k^3),$$

and draw the shear and moment diagrams.

Prob. 5. A load of 1600 pounds is at the middle of the first span and another load of 400 pounds is at the middle of the second span. Show that the reactions are $R_1 = 575$, $R_2 = 1650$, $R_3 = 175$ pounds. Draw the shear and moment diagrams.

ART. 6. REACTIONS FOR THREE SPANS.

Continuous bridges of three spans usually have the end spans equal in length and shorter than the middle span. Continuous draw bridges, on the other hand, often have a middle span which is much shorter than the end spans. Let l be the length of each end span, and nl the length of the middle span, n being a number either greater or less than unity. Let all supports be on the same level.

CASE I, A Load P on End Span. — Let the load P be on the first span at a distance kl from the left support. By the same method as before, the reactions are found to be as follows:

$$R_1 = P(1-k) - \frac{2+2n}{m} P(k-k^3),$$

$$R_2 = Pk + \frac{2+5n+2n^2}{mn} P(k-k^3),$$

$$R_3 = -\frac{2+3n+n^2}{mn} P(k-k^3),$$

$$R_4 = +\frac{n}{m} P(k-k^3),$$

in which, for abbreviation, m represents the quantity

$$4(1+n)^2 - n^2.$$

In Fig. 7, the distribution of shears and moments for this case

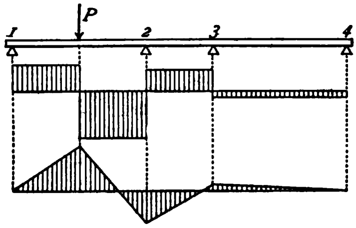

Fig. 7.

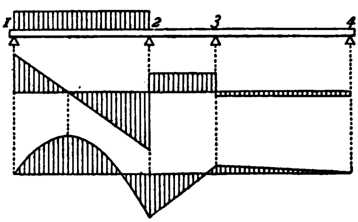

Fig. 8.

is shown. These formulas also give, by changing the subscripts, the reactions for a load on the last span at a distance kl from the support *4*. In computing reactions, the table at the end of this chapter will be of service; for example, if $n = 1$, then $R_3 = -\frac{6}{15}P(k - k^3)$; now, if $k = 0.52$, the table gives $k - k^3 = 0.3794$, and hence $R_3 = -1.5176\,P$.

CASE II, Uniform Load on End Span. — Let the first span be covered with the uniform load wl. Then the reactions are found to be

$$R_1 = \frac{6 + 14\,n + 6\,n^2}{4\,m}\,wl, \qquad R_3 = -\frac{2 + 3\,n + n^2}{4\,mn}\,wl,$$

$$R_2 = \frac{2 + 13\,n + 18\,n^2 + 6\,n^3}{4\,mn}\,wl, \quad R_4 = \frac{n}{4\,m}\,wl,$$

in which m denotes the same quantity as before. If $n = 1$, the three spans are equal and $m = 15$, then $R_1 = +\frac{26}{60}\,wl$, $R_2 = +\frac{39}{60}\,wl$, $R_3 = -\frac{6}{60}\,wl$, $R_4 = +\frac{1}{60}\,wl$. A diagram of shears and moments for this case is seen in Fig. 8.

CASE III, Load P on Middle Span. — Let the load P be in the middle span at a distance $k(nl)$ from the support *2*. From the theorem of three moments, the moments at *2* and *3*, due to this load, are

$$M_2 = -\frac{Pln}{m}\left[(2 + 2\,n)(2\,k - 3\,k^2 + k^3) - n(k - k^3)\right],$$

$$M_3 = -\frac{Pln}{m}\left[(2 + 2\,n)(k - k^3) - n(2\,k - 3\,k^2 + k^3)\right],$$

and from these the reactions are found to be

$$R_1 = \frac{M_2}{l}, \quad R_2 = P(1 - k) - \frac{M_2}{l} + \frac{M_3 - M_2}{nl},$$

$$R_4 = \frac{M_3}{l}, \quad R_3 = Pk - \frac{M_3}{l} - \frac{M_3 - M_2}{nl},$$

in which m has the same signification as in Case I.

CASE IV, Full Uniform Load. — For a load of wl per linear unit over all three spans, the reactions of the supports are

$$R_1 = R_4 = \frac{6 + 15n + 4n^2 - n^3}{4m} wl,$$

$$R_2 = R_3 = \frac{6 + 23n + 18n^2 + 9n^3}{4m} wl,$$

and from these the shears and moments at all sections can be determined. If the spans are of equal length $R_1 = R_4 = \frac{4}{10} wl$, and $R_2 = R_3 = \frac{11}{10} wl$.

Prob. 6. Deduce the formulas for the reactions of a continuous beam of three spans due to a uniform load in the middle span, and draw the shear and moment diagrams. If the spans are equal, show that $R_1 = R_4 = -\frac{1}{20} wl$, and $R_2 = R_4 = +\frac{11}{20} wl$.

ART. 7. LOADINGS FOR MAXIMUM SHEARS.

In order to be able to compute the maximum and minimum stresses in the web members of a continuous truss, it is necessary to know the positions of the live load which give the largest positive and negative shears at any designated section. The shear V is the algebraic sum of all the external forces on the left of the section, and may be either positive or negative according as the reactions on the left of the section exceed or are less than the loads.

TWO SPANS. — For two equal continuous spans the shear at any section in the first span is

$$V = R_1 - \Sigma P,$$

where ΣP denotes the sum of all the loads between the section and the left end. Here the reaction R_1 is positive for all loads on the first span, and negative for all loads on the second span. (Art. 5.) Hence, the largest positive shear at a section in the first or left-hand span, will occur when the live load extends from the section to the middle support, as in Fig. 9. Also the largest negative shear will occur when the live load extends

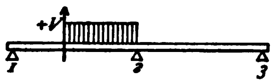

Fig. 9.

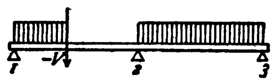

Fig. 10.

from the section to left support, and also covers the second span as in Fig. 10. The live load being placed in these positions and reactions found, the live load shears are computed.

THREE SPANS. — For three continuous spans, the reaction R_1 is positive for all loads in the first and last spans, and negative for all loads in the middle spans, as the formulas in Art. 6 show. The shear for any section in the first span being $R_1 - \Sigma P$, it is easy to see that Fig. 11 shows the arrangement of live load for

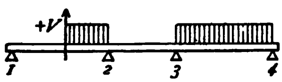

Fig. 11.

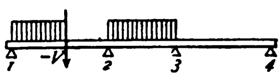

Fig. 12.

largest positive shear, and Fig. 12 that for largest negative shear in the first span.

For a section in the middle span the shear is expressed by

$$V = R_1 + R_2 - \Sigma P_1 - \Sigma P,$$

where ΣP_1 denotes all the live loads on the first span, and ΣP those on the middle span at the left of the section. Here it is seen upon reflection that the largest positive shear occurs when the first span is loaded, and also the segment between the section and support *3*, while the largest negative shear occurs when the last span is loaded, and also the segment between support *2* and the section.

For a continuous truss the loads are applied at the different panel points, and the reactions due to each load being computed the reactions caused by any combination of loads are found by addition, and then the shears are deduced. A numerical example is given in Art. 9.

Prob. 7. What arrangement of live load will give the largest negative value of R_1 for the case of two spans, and also for the case of three spans? What arrangement will give the largest positive value of R_2?

ART. 8. LOADINGS FOR MAXIMUM MOMENTS.

In order to compute the maximum and minimum stresses in the chord members of a continuous truss it is necessary to know the positions of the live load that give the largest positive and negative moments for each member. The bending moment M at any section is the algebraic sum of the moments of the external forces on the left of that section with respect to a point in that section. For a section in the first span l_1 at a distance x from the left support, the moment is

$$M = R_1 x - \Sigma P(x - kl_1),$$

and M may be either positive or negative according as the moment of the reaction is greater or less than the sum of the moments of the loads.

TWO SPANS. — For two equal continuous spans the reaction R_1 is positive for all loads on the first span, and Fig. 4 shows the distribution of moments for a single load. The inflection point is always, as shown in Art. 5, on the one-fifth of the span nearest the support *2*. Thus the moment due to any load P is always positive for a section on the four-fifths of the span nearest *1*, but for a section on the one-fifth nearest *2* it may be either positive or negative. For these two parts of the span there are hence different loadings for maximum moments.

CASE I. — Let i be a point whose distance from the left support is four-fifths of the span. Then every load on the first span causes a positive moment at all sections on the left of i, and every load on the second span causes a negative moment.

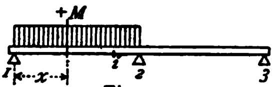

Fig. 13.

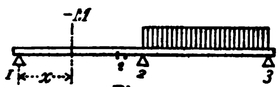

Fig. 14.

Hence the largest positive moment for any section on the left of i occurs when the live load covers the first span, as in Fig. 13; and the largest negative moment occurs when it covers the second span, as in Fig. 14. These are the live load loadings for maximum and minimum moment when x is less than $\frac{4}{5}\,l$.

CASE II. — For a section on the right of i a load P will produce a positive moment if its inflection point is on the right of the section and a negative one if it is on the left. The position of P causing zero moment in the section is given by $R_1x - P(x - kl) = 0$, or inserting for R_1 its value from Art. 5, and replacing k by k_0, there results,

$$k_0 = \sqrt{5 - 4\frac{l}{x}},$$

which gives the limiting positions of the live load. Hence the largest positive moment in the section occurs when the live load extends from the end of the distance k_0l to the support 2,

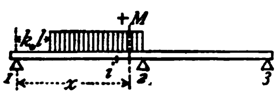

Fig. 15.

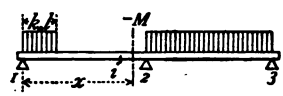

Fig. 16.

as in Fig. 15; and the largest negative moment occurs when the live load covers the distance k_0l and the second span, as in Fig. 16. These are the loadings for maximum and minimum moments when x is greater than $\frac{4}{5}\,l$.

For example, if $x = 0.8\,l$, then $k_0 = 0$ which gives the same loadings as the cases of Figs. 13 and 14. If $x = 0.9\,l$, then $k_0 = 0.745$. If $x = l$, then $k_0 = 1$, and there is no live load for

Fig. 15, while the live load covers the whole truss for Fig. 16; thus the greatest negative moment over the center pier occurs when the bridge is fully loaded.

THREE SPANS. — By a similar method of reasoning the following distributions of live load may be deduced for the different sections. Here, as in Art. 6, the length of each end span is l, and the length of the middle span is nl. The diagrams for the eight different cases should be drawn by the student.

First Span. — For the first span at the left end there is as before a critical inflection point i which divides the span into two parts, and there are different loadings for each part. The formula

$$i = \frac{m}{m + 2n + 2} l$$

gives the distance of this point from the left support, m being $4(1 + n)^2 - n^2$.

When x is less than i, the largest positive moment occurs when the live load covers the two end spans, and the largest negative moment when the live load covers the middle span.

When x is greater than i there is a limiting distance $k_0 l$, as in the case of two spans, and the value of k_0 is

$$k_0 = \sqrt{\frac{m + 2n + 2}{2n + 2} - \frac{m}{2n + 2} \cdot \frac{l}{x}}.$$

Here the largest positive moment occurs when the live load covers the distance from the end of $k_0 l$ to the support 2, and the last span is also loaded. The largest negative moment occurs when the live load covers the distance $k_0 l$ and also the middle span.

Middle Span. — There are two points i_1 and i_2 which divide the span into three parts, and there are different loadings for each part. The distances from the support *2* to these points are

$$i_1 = \frac{n}{2+3n} nl, \qquad i_2 = \frac{2+2n}{2+3n} nl.$$

If the three spans are equal, $n = 1$ and $i_1 = \frac{1}{5} l$, $i_2 = \frac{4}{5} l$.

When x is less than i there is a load limit $k_0(nl)$, and the value of k_0 is one of the roots of the equation

$$[(6n+4n^2)\frac{x}{l} - 3 - 2n]k^2 - [(3n+2n^2)\frac{x}{l} - 3 - 4n]k - m\frac{x}{l} = 0,$$

the largest positive moment occurring when the live load covers the distance $k_0(nl)$ and also the last span, while the largest negative moment occurs when the live load extends from the support *3* to the end of the distance $k_0(nl)$, and also over the first span.

When the section is on the middle part, or when x lies between i_1 and i_2, the largest positive moment occurs for live load over the entire middle span, and the largest negative moment for live load over the two end spans.

When x is greater than i_2 no special rules are necessary, as on account of the symmetry of the girder, computations need be carried no further than the center of the middle span.

Prob. 8. A continuous girder of two equal spans, each 30 feet long, has a dead load of 250 and a live load of 500 pounds per linear foot. Compute the maximum and minimum bending moments at the section in the first span, where $x = 0.8\,l$.

ART. 9. A TWO-SPAN WARREN TRUSS.

A highway deck truss of the Warren type is continuous over three level supports forming two spans, each 50 feet in length. The panel length is 10 feet and the depth of truss 6 feet. The dead load per linear foot per truss is 500 pounds, the snow load 200 pounds, and the live load 1000 pounds, all being taken on the upper chord. It is required to compute the maximum and minimum stresses in all members.

The panel loads and stresses will be expressed in units of one thousand pounds, and this unit will be called a kip, which is an abbreviation for kilo-pound. The dead panel load is 5000

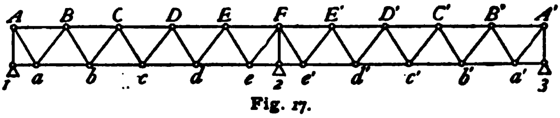

Fig. 17.

pounds, or 5 kips, the snow panel load 2 kips, and the live panel load 10 kips. At the end apexes A and A' the panel loads are one-half these values, but need not be considered except for the end posts.

The chords being horizontal, the stress in any web member is equal to the shear upon it multiplied by the secant of the angle which it makes with the vertical; this secant is

$$\sqrt{5^2 + 6^2}/6 = 1.302.$$

The stress in any chord member is equal to the bending moment divided by the depth of the truss. To obtain the maximum and minimum web stresses, it will be best to compute the maximum and minimum shears and then multiply these by 1.302. Likewise for the chords, the maximum and minimum moments will be found, and then these will be divided by 6 feet. For the live load, the reactions due to each panel load are first computed by the formulas of Art. 5, Case I. For the panel loads at B, C, D, E, the values of k are 0.2, 0.4, 0.6, 0.8, and in finding the values of $k - k^3$ the table in Art. 13 may be used. The reaction R_1 for a load at B' is evidently the same as the reaction R_3 for a load at B, and similarly for loads at C', D', E'. Thus are found

For load at	B	C	D	E	E'	D'	C'	B'
Reaction $R_1 =$	+ 7.52	+ 5.16	+ 3.04	+ 1.28	− 0.72	− 0.96	− 0.84	− 0.48
Reaction $R_2 =$	+ 1.48	+ 2.84	+ 3.96	+ 4.72	+ 4.72	+ 3.96	+ 2.84	+ 1.48

For a load at F, the reactions R_1 and R_3 are 0, and $R_2 = +10$; hence this load produces no stresses in the truss.

The largest positive and negative shears due to live load are now computed, the live load being placed in the positions shown by Figs. 9 and 10 in Art. 7. For instance, for *Cc* the largest positive shear occurs when only the loads D and E are on the truss; the reaction due to these is $+4.32$, and thus $V = +4.32$ kips. The largest negative shear for *Cc* occurs when all loads

SHEARS.	*Aa*	*Bb*	*Cc*	*Dd*	*Ee*
Live load +	+ 17.00	+ 9.48	+ 4.32	+ 1.28	0
Live load −	− 3.00	− 5.48	− 10.32	− 17.28	− 26.00
Total	+ 14.00	+ 4.00	− 6.00	− 16.00	− 26.00
Snow load	+ 2.80	+ 0.80	− 1.20	− 3.20	− 5.20
Dead load	+ 7.00	+ 2.00	− 3.00	− 8.00	− 13.00
Max. shear	+ 28.80	+ 12.28	− 14.52	− 28.48	− 44.20
Min. shear	+ 4.00	− 3.48	+ 1.32	− 6.72	− 13.00
Max. stress	+ 37.5	+ 16.0	− 18.9	− 37.1	− 57.6
Min. stress	+ 5.2	− 4.5	+ 1.7	− 8.7	− 16.9

except D and E are on the truss; the reaction due to these is $+9.68$, and then $V = +9.68 - 20 = -10.32$ kips. The largest live load shears are thus computed and arranged in the first and second lines of the above table. The algebraic sums of these, given in the third line, are the shears due to a uniform live load over the entire truss. One-fifth of these totals give the shears due to snow load, and one-half of them give the shears due to dead load. Then, remembering that the dead load must act, and that the snow and live loads may act either together or separately, the maximum and minimum shears are found. These multiplied by 1.302 give the maximum and minimum stresses in kips, + denoting tension and − denoting compression. The stresses for the members *Ba, Cb, Dc, Ed, Fe* are the same as those for *Aa, Bb, Cc, Dd, Ee*, respectively, but with contrary signs. The signs of the stresses will generally be apparent upon reflection, but if not they can be found from the signs of the shears by the following rule:

> A positive shear causes $\left\{\begin{matrix}\text{tension}\\\text{compression}\end{matrix}\right\}$ in a diagonal sloping $\left\{\begin{matrix}\text{downward}\\\text{upward}\end{matrix}\right\}$ away from the left-hand abutment; a negative shear causes the reverse.

For the chord stresses a similar method of tabulation will be used. The largest possible moment for *CD* due to live load is when the first span is fully loaded, as in Fig. 13 of Art. 8; the reaction R_1 due to these loads is $+ 17.00$, and $M = 17 \times 25 - 10(15 + 5) = + 225$ kip-feet. For *EF* the center of moments is at *e*, which is more than four-fifths the span, and thus the loading is as in Fig. 14; here $k_0 = 0.74$, which shows that only the load *E* should be on the truss for largest positive moment, and

MOMENTS.	*AB*	*CD*	*EF*	*ab*	*cd*	*e2*
Live load +	+ 85	+ 225	+ 8	+ 170	+ 210	0
Live load –	– 15	– 75	– 178	– 30	– 90	– 300
Total	+ 70	+ 150	– 170	+ 140	+ 120	– 300
Snow load	+ 14	+ 30	– 34	+ 28	+ 24	– 60
Dead load	+ 35	+ 75	– 85	+ 70	+ 60	– 150
Max. moment	+ 134	+ 325	– 297	+ 268	+ 294	– 510
Min. moment	+ 20	0	– 77	+ 40	– 30	– 150
Max. stress	– 22.3	– 54.2	+ 49.5	+ 44.7	+ 49.0	– 85.0
Min. stress	– 3.3	0	+ 12.8	+ 6.7	+ 5.0	– 25.0

that all loads except *E* are on the truss for largest negative moment. For *e2* both spans fully loaded give the largest negative moment. The live load moments being computed are arranged in the first and second lines of the above table. The algebraic sums of these give the moments due to live load over the entire span. One-fifth of these totals give the moments due to snow load, and one-half give the moments due to dead load. Then, noting that dead load always acts, and that live and snow loads may act, the maximum and minimum moments are readily found. These are in kip-feet, and dividing them by 6 feet, the maximum and minimum chord stresses result. The signs of these stresses may always be found from the signs of the moments by the following rule:

A positive moment causes $\left\{ \begin{matrix} \text{tension} \\ \text{compression} \end{matrix} \right\}$ in the $\left\{ \begin{matrix} \text{lower} \\ \text{upper} \end{matrix} \right\}$ chord, while a negative moment causes the reverse.

Prob. 9. Compute the maximum and minimum stresses in the members *BC*, *DE*, *1 a*, *bc*, *de*, *1 A*, *2 F*, for the above data.

ART. 10. A THREE-SPAN PRATT TRUSS.

A railroad continuous truss has three spans, the end ones being 75 feet and the middle one 18 feet. The middle span consists of one panel 21 feet in height. In each end span there are five panels, 15 feet long, while the depths of the truss at *A*, *B*, *C*, *D*, *E* are 17, 18, 19, 20, 21 feet, respectively. The

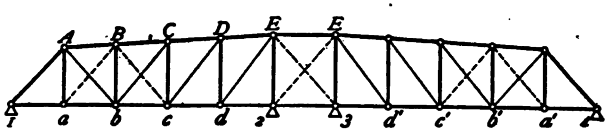

Fig. 18.

dead load per linear foot of track is 1200 pounds, of which three-fourths is on the lower chord. The live load is 3600 pounds per linear foot of track, which is regarded as an equivalent for full locomotive load. The bridge having two tracks, the dead panel load is 18 kips, of which $4\frac{1}{2}$ kips is on the upper chord, while the live panel load is 54 kips, one kip being 1000 pounds. The truss being of the Pratt type, the diagonals can take only tension, and the broken diagonals are those not stressed under dead load.

The reactions due to the live loads are first to be computed by the formulas of Art. 6, Case I. Here $n = 18/75 = 0.24$, $m = 6.0928$ or $1/m = 0.16413$. Then, as P is 54 kips, the formulas become

$$R_1 = 54.00(1-k) - 21.98(k-k^3), \quad R_3 = -102.57(k-k^3),$$
$$R_2 = 54.00\,k - 75.98(k-k^3), \quad R_4 = +2.13(k-k^3).$$

In computing the reactions for different values of k the table in Art. 13 will be found useful. For a load at a the value of k is 0.2, then $R_1 = +38.98$, $R_2 = +34.30$, $R_3 = -19.69$, $R_4 = +0.41$; the reactions for a load at a' are the same as these in reverse order. The reactions R_1 and R_2 for each load are, in kips,

Load at	a	b	c	d	d'	c'	b'	a'
Reaction $R_1=$	+38.98	+25.01	+13.16	+ 4.47	+ 0.61	+ 0.82	+ 0.72	+ 0.41
Reaction $R_2=$	+34.30	+62.73	+79.41	+78.46	−29.54	−39.39	−34.46	−19.69

The sums of these give the reactions $R_1 = +84.18$, $R_2 = +131.82$, for a uniform live load. Since the dead load is one-third the live load, $R_1 = +28.06$ and $R_2 = +43.94$ are the dead load reactions.

As this truss has an inclined upper chord the method of the last article must be modified for the web members. The method of moments will be used for all members except the end post and the diagonals in the middle span. A section being passed cutting a member and two others, the center of moments for that member is at the intersection of the other two. To abbreviate the work the panel length will be called p and all lever arms be expressed in terms of p; thus for cd the center of moments is at D and the lever arm is $\frac{20}{15}p$; for Cc the center of moments is on the line of the lower chord at a distance $16p$ to the left of the support *1*, and its lever arm is $19p$. When the equation of moments for a member is stated, p appears in each term and cancels out. In writing the equations it is to be noted that the full dead load always acts, but the live loads to be used for a web or chord member will be those giving the largest shear or moment, as deduced in Arts. 7 and 9.

For the main diagonal *Dc* the maximum tension S occurs under the largest negative and the minimum tension S' under the largest positive shear. The first happens when the live loads a, b, c, are on the truss, and the second when all live loads except these are on the truss; the live load reaction R_1 is $+77.15$ for the first case and $+7.03$ for the second. The center of moments for *Dc* is at the point where the two chords intersect and the lever arm is $15.2p$. The equations of moments for the two cases are

$$-(77.15+28.06)16p+(54+18)(17p+18p+19p)-S\times 15.2p=0,$$
$$-(7.03+28.06)16p+18(17p+18p+19p)-S'\times 15.2p=0,$$

whence $S=+145$ and $S'=+27$ kips are the maximum and minimum tensions for *Dc*.

In the same manner for *Cb* the equations of moments are

$$-(63.99+28.06)16p+72(17p+18p)-S\times 14.13p=0,$$
$$-(20.19+28.06)16p+18(17p+18p)-S'\times 14.13p=0,$$

whence $S=+74$ and S' is negative. But as a diagonal cannot take compression the counter diagonal *Bc* is needed in this panel. The maximum tension for *Bc* is given by the last equation, replacing the last term by $+S'\times 13.5p$, from which $S'=+10$, while the minimum tension for both *Cb* and *Bc* is zero.

The following are the stresses for the diagonals in kips:

	A1	*Ba*	*Cb*	*Dc*	*Ed*	*E2*	*Ab*	*Bc*
Maximum	− 150	+ 12	+ 74	+ 145	+ 220	+ 159	+ 64	+ 10
Minimum	− 37	0	0	+ 27	+ 52	0	0	0

The stresses for the verticals are found in a similar manner. For *Dd* a section is passed cutting it and the two chords, and the upper panel load at *d* is thus on the left of the section. The center of moments is at the point where the chords intersect, and the two equations of moments are

$$-105.21 \times 16p + 72(17p + 18p + 19p) + 4\tfrac{1}{2} \times 20p + S \times 20p = 0,$$
$$-35.09 \times 16p + 18(17p + 18p + 19p) + 4\tfrac{1}{2} \times 20p + S' \times 20p = 0,$$

whence $S = -115$ and $S' = -25$ kips. For *Bb* and *Cc* the minimum stress is simply the upper panel load. For *Aa* the stress is always tension, the minimum being the lower panel dead load. The final stresses for the verticals, in kips, are then as follows:

	Aa	*Bb*	*Cc*	*Dd*	*E2*
Maximum	+ 37	− 13	− 60	− 115	− 178
Minimum	+ 13	− 4	− 4	− 25	− 41

The positions of the live load causing the maximum and minimum chord stresses in the first span are stated in Art. 9. The critical point *i* is found to be 53.3 feet from the left support or in the panel *cd*, and for all centers of moments preceding this the largest positive moment occurs when the two side spans are fully loaded. The reaction R_1 due to dead and live loads for this case is + 112.24, the shear in the second panel is + 30.24 and the shear in the third panel is − 41.76; the diagonals *Ab* and *Cb* are hence in action. The center of moments for the upper chords *AB* and *BC* is then at *b*, and the equation of moments is

$$+ 112.24 \times 2p - 72 \times p + S \times 1.197p = 0,$$

whence $S = -127$ kips is the maximum compression in *AC*, while the minimum which occurs under dead load is one-third of this.

For the chord *cd* whose center of moments lies beyond *i*, the distance x to the center of moments is 60 feet or $l/x = 1\frac{1}{4}$; then $k_0 = 0.62$, so that the loads *a, b, c* constitute one system and the remainder of the live loads the other system. The equations of moments for the two cases are

$$(77.15 + 28.06)4p - 72(3p + 2p + p) - S \times \tfrac{20}{15}p = 0,$$
$$(7.03 + 28.06)4p - 18(3p + 2p + p) - S' \times \tfrac{20}{15}p = 0,$$

whence $-$ 8.3 and + 24.3 are the two limiting stresses for *cd*. The same equations with a different lever arm give the stresses for *DE*. The maximum stresses in *d2*, *23*, *EE'*, occur when the truss is fully loaded.

The following are the final stresses for the lower chords:

	1a	*ab*	*bc*	*cd*	*d2*	*23*
Maximum	+ 99	+ 99	+ 95	+ 24	− 113	+ 113
Minimum	+ 23	+ 23	+ 24	− 8	− 28	− 28

It will be noticed that the middle span of this bridge is very short, so that an effective negative reaction of 79 kips may occur at *2* or *3* when the last or first span is fully loaded. The truss must hence be fastened down at these supports, so that these negative reactions may take effect and produce tension in the verticals $E2$ and $3E'$; if this is not done the truss may rise and become one of two spans under partial live load. This truss is, in fact, that of a swing draw bridge, and it will be further discussed in Art. 18. For a fixed continuous truss the middle span should be slightly longer than the side spans in order to give the best conditions for economy of material.

Prob. 10. Check several of the above stresses, and also compute those for the upper chord. Show that the maximum tension for the middle diagonals $E2$ and $3E'$ is 104 kips.

ART. 11. SUPPORTS ON DIFFERENT LEVELS.

Let an unloaded continuous girder, without weight, and originally straight, have its supports on different levels. The theorem of three moments in Art. 4 becomes

$$M_2 l_2 + 2 M_3 (l_2 + l_3) + M_4 l_3 = -6 EI \left(\frac{h_3 - h_2}{l_2} + \frac{h_3 - h_4}{l_3} \right)$$

where h_2, h_3, h_4, denote the elevations of the supports above an assumed datum plane. If the weightless beam have stiffness, represented by E, the theorem shows that moments obtain at the support, and hence stresses occur at every section. This is because exterior forces are required at the supports to make the elastic curve pass through those points of supports. Thus a support must not only prevent the beam from falling but from rising, in order that the theory of continuity developed in Art. 4 may be valid.

Practically no girder is without weight, and this usually serves to hold it down at the supports under a partial live load. If, however, a negative reaction can occur at any support, the girder must be anchored to that support so as to prevent it rising, and the maximum stress on the anchor rod is equal to the greatest negative reaction. In the last article a practical case of this kind is seen.

If a girder be built on level supports, and one or more of them be lowered, the moments at the supports due to such depressions may be found by the above theorem. For example, let a girder of four equal spans have its center pier lowered a small distance h below the level of the others, and let the girder still touch all the supports. The theorem of three moments, written for the supports *2*, *3*, *4*, gives three equations containing the three unknown moments at these supports; thus,

$$4 M_2 + M_3 = -\frac{6 EIh}{l^2},$$

$$M_2 + 4M_3 + M_4 = +\frac{12\,EIh}{l^2},$$

$$M_3 + 4M_4 = -\frac{6\,EIh}{l^2},$$

from which the moments at the supports are

$$M_2 = M_4 = -\frac{18\,EIh}{l^2}, \qquad M_3 = +\frac{30\,EIh}{7\,l^2},$$

and the reactions are

$$R_1 = R_5 = -\frac{18\,EIh}{l^3}, \quad R_2 = R_4 = +\frac{66\,EIh}{l^3}, \quad R_3 = -\frac{60\,EIh}{l^3},$$

and from these the stresses due to depression h may be computed. Let the span l be 100 feet, the girder a truss whose depth is 10 feet, and chord section 0.42 square feet, let $E =$ 4 320 000 000 pounds per square foot, and $h = 0.05$ feet. Then the moment of inertia of the chord sections is $I = 21$ feet^4, and $M_2 = -388\,800$ pound-feet which is the increase in the moments at the supports *2* and *4*, and $R_1 = -3888$ pounds which is the negative reaction due to the depression of the middle support by 0.05 feet. The upper chord stresses at *2* and *4* are hence increased by 38 900 pounds and the lower chord stresses decreased by the same amount. If the middle support be lowered 0.1 feet the change in these chord stresses will be 77 800 pounds.

It is thus seen that a slight change in level of one of the supports may produce great changes in the stresses in all parts of the girder, and that such changes in level are liable to injure or even to cause the destruction of the girder. This is the strongest objection to the use of fixed continuous spans.

Prob. 11. A continuous girder of two equal spans is loaded uniformly with w per linear foot. If the reaction at each end is $\frac{3}{4}\,wl$, find how far the middle support is depressed below the end supports.

ART. 12. ADVANTAGES AND DISADVANTAGES.

When a bridge is to be built across a river and several spans are required, it is possible to erect either a continuous structure or several simple ones. The arguments in favor of the continuous system are as follows:

1. The simplicity of construction over the piers is greater, since portals are not needed, and when the piers are skewed many expensive details are avoided.

2. The deflection under live load is less, and the stiffness is greater than for simple spans, the injurious effect of oscillation being thus diminished.

3. When false works are difficult or expensive, a continuous truss may be built out from the shore, panel by panel, as with cantilevers.

4. The upper part of the piers may be somewhat smaller for the continuous system, since less bearing surface is required than for the two ends of simple spans.

5. The amount of material required for the continuous system is less than for the simple spans, the saving in material being often as great as twenty-five per cent. This saving occurs mostly in the chords, and is due to the smaller bending moments.

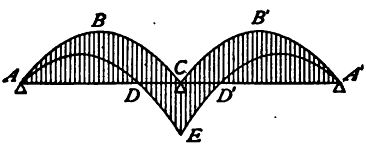

Fig. 19.

Thus in Fig. 19, if AC and CA' be two simple spans, the bending moments due to uniform load are always positive, and are shown by the parabolas ABC and $CB'A'$. If a continuous truss AA' be used, the bending moments are part positive and part negative, and are shown by the parabolas ADE and $ED'A'$. If l be the span and w the load per linear unit, reaction R_1 is $\frac{1}{2}wl$ for the first case and $\frac{3}{8}wl$

for the second. The areas included between the parabolas and the base AC are then

$$A_1 = \int_0^l (R_1x - \tfrac{1}{2}wx^2)dx = \tfrac{1}{12}wl^3 = \tfrac{32}{384}wl^3,$$

$$A_2 = \int_0^{\frac{3}{4}l} (R_1x - \tfrac{1}{2}wx^2)dx - \int_{\frac{3}{4}l}^{l} (R_1x - \tfrac{1}{2}wx^2)dx = \tfrac{19}{384}wl^3.$$

The area included between ABC and AC is hence $\frac{32}{19}$ths of those between ADE and AC. If now, it were possible to design the chords so that their sections are proportional to the bending moments, then the chords of the simple truss would contain $1\frac{13}{19}$ times as much material as the continuous one. It is, however, not practicable to do this, and in fact for short spans it is best to make the chords of uniform section throughout.

The objections to the continuous system for truss bridges may be summarized as follows:

1. The theory is not strictly correct, as it supposes the moment of inertia I to be constant, whereas in a truss it is subject to variation. The error due to this cause rarely gives errors in stresses greater than six per cent.

2. Many of the chord members are subject to alternating stresses of tension and compression which require low unit-stresses to resist them, and hence the saving in material is much less than pure theory indicates.

3. The computation of stresses is much more difficult than for simple trusses, and the erection is made by building out panel by panel; additional computations are needed, and extra material required to resist the erection stresses.

4. Changes of level in piers and abutments cause great changes or reversals of stress. This objection, as shown in Art. 11, is a very serious one, and by reason of it, more than all

others, the continuous system is little used except for draw bridges. It is, however, not important that the piers should be exactly on the same level when the bridge is built, provided that the false works be so arranged that the profile of the unstrained truss agrees exactly with that of the piers.

The longest and almost the only continuous truss bridge in America, exclusive of draw bridges, is the Lachine bridge over the St. Lawrence river near Montreal, built in 1887 by the Dominion Bridge Company, It has four spans, the two side spans being 269 feet each and the two others 408 feet each. It presents a beautiful appearance, as the side spans are deck and the others through, the transition being made by graceful curves. For description of the method of computation and erection, see Engineering News, Oct. 1, 8, and 15, 1887.

Prob. 12. Find the maximum bending moments in the Lachine truss, the dead load being 1300 pounds per linear foot on the side spans and 1650 on the central spans, while the live load is 1500 pounds per linear foot.

ART. 13. GENERAL FORMULAS.

The following general formulas for the moments of continuous girders on level supports were deduced by MERRIMAN from the theorem of three moments and first published in the London Philosophical Magazine for September, 1875.

Let the number of spans be s and the supports be numbered $1, 2, 3, \cdots s, s+1$, as in Fig. 20, the lengths of the spans being

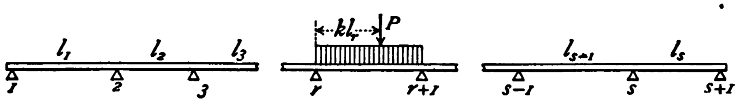

Fig. 20.

$l_1, l_2, \cdots l_s$. Let the span l_r have the single load P and the uniform load wl_r, and let it be required to find the moments at the supports due to these loads.

Let $c_1, c_2, \cdots c_{s+1}$ and $d_1, d_2, \cdots d_{s+1}$ be two series of numbers depending only on the lengths of spans as follows:

$$c_1 = 0,\quad c_2 = +1, \qquad\qquad d_1 = 0,\quad d_2 = +1,$$

$$c_3 = -2 - 2\frac{l_1}{l_2}, \qquad\qquad d_3 = -2 - 2\frac{l_s}{l_{s-1}},$$

$$c_4 = -2c_3 - (2c_3 + c_2)\frac{l_2}{l_3}, \qquad\qquad d_4 = -2d_3 - (2d_3 + d_2)\frac{l_{s-1}}{l_{s-2}},$$

$$c_5 = -2c_4 - (2c_4 + c_3)\frac{l_3}{l_4}, \qquad\qquad d_5 = -2d_4 - (2d_4 + d_3)\frac{l_{s-2}}{l_{s-3}},$$

$$\cdots\cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots\cdots$$

$$c_{s+1}l_s = -2c_s l_s - (2c_s + c_{s-1})l_{s-1}. \qquad\qquad d_s l_1 = -2d_s l_1 - (2d_s + d_{s-1})l_2.$$

Let A and B be quantities depending on the given loads, namely,

$$A = Pl_r^2(2k - 3k + k^3) + \tfrac{1}{4}wl_r^2, \qquad B = Pl_r^2(k - k^3) + \tfrac{1}{4}wl_r^3.$$

Then for any support n the formulas are

when $n < r + 1$, $$M_n = \frac{c_n}{d_{s+1}l_1}(d_{s-r+2}A + d_{s-r+1}B), \qquad (1)$$

when $n > r$, $$M_n = \frac{d_{s-n+2}}{c_{s+1}l_s}(c_r A + c_{r+1}B), \qquad (2)$$

the first giving the moment at n due to the loads in the span l_r for all supports on the left of l_r, and the second for all supports on the right of l_r.

For example, take the case of four spans, where $l_1 = l_4 = l$, $l_2 = l_3 = \frac{4}{3}l$. Here, the series for c and d give:

$$c_2 = +1, \qquad c_3 = -3.5, \qquad c_4 = +13, \qquad c_5 l_4 = -56l,$$

$$d_2 = +1, \qquad d_3 = -3.5, \qquad d_4 = +13, \qquad d_5 l_1 = -56l.$$

Now let it be required to find the moments at the supports due to a load P in the second span. Then $r = 2$ and $s = 4$, and making $n = 2$, the formula (1) gives,

$$M_2 = -\frac{13A - 3.5B}{56l} = -\frac{13Pl}{56}(2k - 3k^2 + k^3) + \frac{3.5Pl}{56}(k - k^3).$$

Again, making $n=4$, formula (2) gives,

$$M_4 = \frac{A - 3.5\,B}{56\,l} = -\frac{Pl}{56}(2\,k - 3\,k^2 + k^3) + \frac{3.5\,Pl}{56}(k - k^3).$$

And again, making $n = 3$ in (2), the moment M_3 is $-3.5\,M_4$. Now for any value of k the moments are readily computed by the help of the table given below; thus for $k = 0.47$ the table gives $(2\,k - 3\,k^2 + k^3) = 0.3811$ and $(k - k^3) = 0.3662$, whence $M_2 = -0.06558\,Pl$, $M_3 = -0.05603\,Pl$ and $M_4 = +0.01601\,Pl$.

The above formulas apply also to continuous girders with ends fixed horizontally. If the left end is fixed, make $l_1 = 0$ and let $s - 1$ be the number of spans, then M_2 is the moment at the left end. If both ends are fixed, make $l_1 = 0$, $l_s = 0$, and let $s - 2$ be the number of spans, then M_2 and M_{s-1} are the moments at the fixed ends. For example, take the case of two equal spans with both ends fixed. Here $c_2 = d_2 = +1$, $c_3 = d_3 = -2$, $c_4 = d_4 = +7$, $c_5 l_4 = d_5 l_1 = -12\,l$. Now let the first span l_2 be covered with the uniform load wl. Then from (1) the moment at the left fixed end is $M_2 = -\frac{5}{48}\,wl^2$, from (2) that at the middle support is $M_3 = -\frac{1}{24}\,wl^2$ and that at the right fixed end is $M_4 = +\frac{1}{48}\,wl^2$.

For any unloaded span the shear V_n at the left end of that span is given by

$$V_n = \frac{M_{n+1} - M_n}{l_n}, \tag{3}$$

and the shear and moment at any section distant x from the support n are found by

$$V = V_n, \qquad M = M_n + V_n x, \tag{4}$$

and thus the stresses in all unloaded spans due to the loads in l_r may be computed.

For the loaded span l_r the shear V_r at the right of the support r is given by

$$V_r = \frac{M_{r+1} - M_r}{l_r} + P(1 - k) + \tfrac{1}{2}\,wl,$$

and the shear at any section distant x from r is

$$\text{for } x < kl_r, \qquad V = V_n - wx, \qquad (5)$$
$$\text{for } x > kl_r, \qquad V = V_n - P - wx;$$

while the moment for any section is

$$\text{for } x < kl_r, \qquad M = M_n + V_n x - \tfrac{1}{2} wx^2, \qquad (5)'$$
$$\text{for } x > kl_r, \qquad M = M_n + V_n x - P(x - kl_r) - \tfrac{1}{2} wx^2,$$

and thus the stresses in the loaded span are found.

By the successive application of the formulas of this article the complete investigation of all continuous girders on level supports is possible. If the girder is horizontally restrained or fixed at both ends, they will also apply by making $l_1 = 0$, $l_s = 0$, and letting $s - 2$ represent the number of spans.

VALUES OF $(k - k^3)$ AND $(2k - 3k^2 + k^3)$.

Read down for $(k - k^3)$.

	0	1	2	3	4	5	6	7	8	9		
0	.0000	0100	0200	0300	0399	0499	0598	0697	0795	0893	0990	9
1	.0990	1087	1183	1278	1373	1466	1559	1651	1742	1831	1920	8
2	.1920	2007	2094	2178	2262	2344	2424	2503	2580	2656	2730	7
3	.2730	2802	2872	2941	3007	3071	3134	3193	3251	3307	3360	6
4	.3360	3411	3459	3505	3548	3589	3627	3662	3694	3724	3750	5
5	.3750	3773	3794	3811	3825	3836	3844	3848	3849	3846	3840	4
6	.3840	3830	3817	3800	3779	3754	3725	3692	3656	3615	3570	3
7	.3570	3521	3468	3410	3348	3281	3210	3135	3054	2970	2880	2
8	.2880	2786	2686	2582	2473	2359	2239	2115	1985	1850	1710	1
9	.1710	1564	1413	1256	1094	0926	0753	0573	0388	0197	0000	0
		9	8	7	6	5	4	3	2	1	0	

Read up for $(2k - 3k^2 + k^3)$.

CHAPTER II.

DRAW BRIDGES.

ART. 14. CLASSIFICATION.

Under the general term "draw bridges" are here included all structures over rivers that can be moved in order to secure a clear passage way for boats. The ancient draw bridge which spanned a moat around a castle usually turned on hinges at the inside of the moat, and was pulled up or let down by a chain; it embodied the general ideas of the hinged lift structure shown in Fig. 41. Another old form of draw bridge was rolled on wheels back from the moat, the inside end being usually weighted to insure stability.

Modern draw bridges may be classified as swing bridges, rolling bridges, and lift bridges, the first being the most common type. A swing bridge is supported upon a pier at the middle, and when closed the ends rest upon abutments. When open each arm is a cantilever; when closed the structure may be arranged to form two simple spans, or to be continuous over all the supports.

Rolling draw bridges are those which have wheels under the land portions, and which can be pushed out to span the stream. In some cases the structure consists of two parts, one on each shore, and the water ends of these are locked together when the bridge is closed. Rolling draw bridges are used but little.

Lift bridges are of various kinds. The simplest is a common truss which is raised vertically to the desired height, both ends rising in guides arranged on towers. The hinged lift bridge moves in a vertical plane around hinges at one end, like the ancient draw over the castle moat. The rolling lift bridge is raised in a similar manner, but also has a slight rolling motion.

An attendant is always necessary to operate a draw bridge. The smaller and lighter structures are moved by man power, the others by steam or electric power. A swing bridge rests upon a turntable which is revolved by a rack and pinion arrangement. A rolling bridge is pulled back by the rope and drum method. A lift bridge usually has a counterweight to assist the motion. When land and water traffic is heavy it is necessary that the structure should move quickly, one minute being frequently specified as the time of opening or closing.

Prob. 13. See Engineering News, Oct. 27, 1892, for thirteen designs proposed for a draw bridge over the Duluth ship canal; ascertain which of these, if any, was built at that location. See also Engineering News, Nov. 5, 1896, for designs presented in the Newtown Creek bridge competition.

ART. 15. SWING BRIDGES.

The old form of swing bridge had a tower over the center pier, from which inclined chains extended to the ends. These ends rested upon the abutments loosely, so that a live load upon one span lifted the other end. The arrangement was a bad one in all respects, and is now never used. The modern continuous swing bridge is of the type shown in Fig. 35, and when it is closed the ends are locked so as to secure full continuity and prevent injurious oscillations.

Another method of arranging the ends is to lift them by wedges as soon as the bridge is closed, thus causing reactions under dead load. The dead load stresses are then governed by the principles of continuity. The method is an objectionable one because these reactions depend upon the amount of lift and there is no exact method of determining them.

The usual method of arranging the ends is to lock them by bolts or pins which produce no reactions when the bridge is unloaded. The dead load stresses are hence the same whether the bridge be open or closed, but when the live load comes upon the truss the locking pins may take either positive or negative reactions, and accordingly the live load stresses are governed by the laws of continuous girders set forth in Chapter I.

There are hence three methods of arranging the ends of a swing bridge, loose ends, lifted ends, locked ends. The first should never be used, the second should be avoided or used with great caution, while the third method may be safely employed as reliable both in theory and practice.

The draw bridge trusses are generally arranged so as to rest upon supports over the pier. These supports rest upon the turntable, which in turn is supported upon a pivot or upon a series of wheels that enable it to be turned upon the pier. There are two methods of supporting the turntable; the first shown diagrammatically in Fig. 21 is the center-bearing method

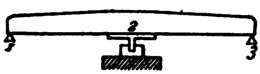

Fig. 21.

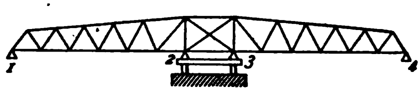

Fig. 22.

used for short spans where the entire weight is carried by a central pivot. The second and common method is the rim-bearing method where the weight is carried upon a series of wheels around the circumference of the turntable. A center-bearing swing bridge is a continuous truss of two spans, as the turntable is really a part of the bridge. A rim-bearing bridge is a truss of three spans. In each case the beams which connect the trusses with the turntable must be securely fastened to both.

In the center-bearing method the entire weight may be carried on a pivot resting in a step, as indicated in Fig. 21, or the pivot may be supported by balls or by conical rollers. In the rim-bearing method wheels are used instead of balls, and the turntable is moved by the help of a rack and pinion. Connected with the apparatus that moves the bridge are levers and rods which automatically open and shut the locking bolts at the ends, and also hoist and lower the danger signals.

Swing bridges which have the center-bearing arrangement of Fig. 21 are necessarily continuous and of two spans; they are often built as plate girders. Those with the rim-bearing turntable are properly considered as of three spans, and the trusses may be continuous, partially continuous, or simple; these three classes will be discussed in the following articles.

The longest swing bridge is that designed by J. A. L. WADDELL, and erected in 1893 over the Missouri river at Omaha, Neb., its length being 520 feet; see Engineering News, Dec. 7, 1893. The heaviest is that designed by A. P. BOLLER, and erected in 1895 over the Harlem river at New York, there being three parallel trusses carrying four railroad tracks; see Railroad Gazette, Feb. 21, 1896.

Prob. 14. Sketch a ball-bearing pivot for a locomotive turntable (see Engineering News, April 1, 1897).

Prob. 15. If each span in Fig. 21 is 50 feet long, find the greatest negative reaction due to a live load of 2000 pounds per linear foot.

ART. 16. A CENTER-BEARING CONTINUOUS TRUSS.

A highway deck swing bridge of the Warren type is continuous over a center-bearing pivot, and has locked ends. It is 100 feet long, has 10 panels, and is 6 feet deep. The dead load per linear foot per truss is 500 pounds, the snow load 200 pounds, and the live load 1000 pounds, all on the upper chord. The dead panel load is 5000 pounds, or 5 kips, the snow panel

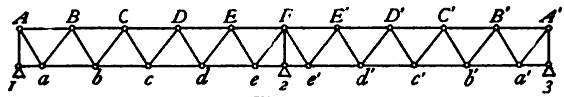

Fig. 23.

load 2 kips, and the live panel load 10 kips; at the end apexes A and A' the panel loads are one-half these values. It is required to compute the maximum and minimum stresses due to these loads.

The dead load stresses are the same when the bridge is shut as when it is open, the reactions at *1* and *2* being then zero. Thus for the members Cc and Dc the dead load shear is $-2.5 - 5 - 5 = -12.5$ kips, this producing compression in Cc and tension in Dc. For the chord bc the bending moment is $-2.5 \times 25 - 5(15 + 5) = -162.5$ kip-feet, which gives compression in bc. The shears multiplied by sec θ, or 1.302, give the web stresses, and the moments divided by 6 feet give the chord stresses.

The snow load is properly considered as upon the bridge only when it is shut, since it could not be closed if snow were upon it when open. The reactions due to snow hence follow the law of continuity, and the snow stresses are found by taking one-fifth of those produced by a uniform live load over the whole bridge.

For the live load the reactions due to each panel load are found by Art. 5, and then by the method of Art. 9 the greatest live load shears and moments are computed. As the data here given are the same as those in Art. 9, the live load shears and moments there tabulated may be directly used, the former being multiplied by 1.302 and the latter divided by 6 to give the live load stresses as tabulated below:

DIAGONALS.	*Aa*	*Bb*	*Cc*	*Dd*	*Ee*
Live load $+V$	+22.13	+12.34	+5.62	+1.67	0
Live load $-V$	−3.90	−7.14	−13.44	−22.50	−33.85
Snow load	+3 65	+1.04	−1.56	−3.12	−6.77
Dead load	−3.25	−9.77	−16.27	−22.79	−29.28
Max. stress	+22.5	−16.9	−31.3	−48.4	−69.9
Min. stress	−7.2	+3.6	−10.6	−20.1	−29.3

The algebraic sum of the + and − live load stresses are the live load stresses due to a uniform live load, and one-fifth of these sums gives the snow load stresses. Then remembering

CHORDS.	*AB*	*CD*	*EF*	*ab*	*cd*	*ef*
Live load $+M$	−14.17	−37.50	−1.33	+28.33	+35.00	0
Live load $-M$	+2.50	+12.50	−29.67	−5.00	−15.00	−50.00
Snow load	−2.33	−5.00	+5.67	+4.67	+4.00	−10.00
Dead load	+2.08	+27.08	+53.75	−10.42	−37.50	−104.17
Max. stress	−14.4	+39.6	+59.4	+22.6	−52.5	−164.2
Min. stress	+4.6	−5.4	+24.1	−15.4	+1.5	−104.2

that the dead load always acts, and that the snow and live loads may act either together or separately, the maximum and minimum stresses are found. These are in kips, one kip being 1000 pounds.

Prob. 16. If this truss have lifted ends, and if the amount of lift be such that the dead load stresses follow the law of continuity when the bridge is shut, show that the maximum and minimum stresses for *Aa*, *AB*, *ab* are + 34.9 and − 3.3, − 22.3 and + 2.1, + 44.7 and − 10.4 kips.

ART. 17. PLATE GIRDER SWING BRIDGES.

Plate girders are used for deck swing bridges with lengths up to nearly 200 feet. Being built shallower than trusses, they are stiffer under the passage of traffic. They are simple in construction, are quickly erected, and when steel is cheap are often more economical than lattice girders or pin trusses. The ends may be locked so that the dead load stresses are the same whether the bridge be open or closed, or they may be lifted so that the dead load stresses follow the law of continuity when the bridge is closed, the former being the preferable method. The depth of the girder is usually made less near the ends than near the middle, as this is conducive to uniformity of flange sections. The live load to be used in computing the shears and moments may be a heavy uniform load or actual locomotive wheels.

The principles in Arts. 7 and 8 will serve to show how the live load should be placed to give the maximum shears and moments at different sections. Let the live load consist of the typical consolidation locomotives shown in Fig. 24, which is class *T* of the compromise standard system recommended by WADDELL. For maximum shear at the left end of the girder,

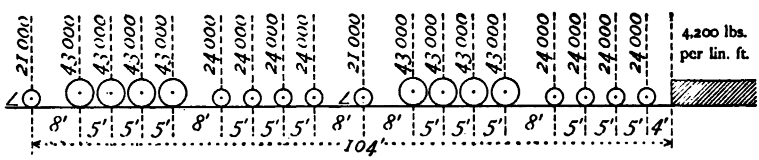

Fig. 24.

the load should come on from the left until the last driver is just entering upon the span. For maximum shear at the pivot pier, the load should pass on until the first driver is nearly at the pier. For maximum positive moment in the first span, that span should be as fully loaded as possible with the center of gravity of the drivers on the left of the center of the span. For maximum negative moment at the first pier, both spans should be fully loaded, with the locomotives facing each other on the two spans, and as near the pier as possible. Trial will be necessary to determine the exact position, but after a little practice it can be closely assigned at the outset.

For example, let the length of each span be 50 feet. Then Fig. 25 shows the position of the wheels for maximum positive shear at the left end, and Fig. 26 that for maximum negative

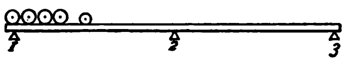

Fig. 25.

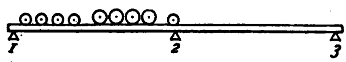

Fig. 26.

shear at the middle. Also Fig. 27 gives the position for maximum positive moment, and Fig. 28 that for maximum negative

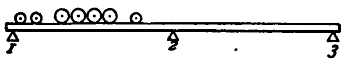

Fig. 27.

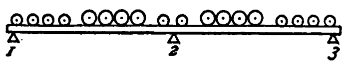

Fig. 28.

moment. Let the bridge have two tracks; then the loads in Fig. 24 are the wheel loads for one girder. In what follows, these loads will be divided by 1000, and thus be expressed in kips.

In Fig. 25 the distance of the pilot wheel from the left end is 23 feet, and those of the drivers 15, 10, 5, 0 feet. Thus, the values of k are 0.46, 0.30, 0.20, 0.10, 0.00. The reaction R_1 due to each of these loads is computed by the formula of Art. 5, and the sum of these reactions is 149.6 kips, or 149 600 pounds, which is the maximum positive shear at the left end. For Fig. 26, placing the pilot wheel at the support *2*, the values of k are 1.00, 0.84, 0.74, 0.64, 0.54, 0.38, 0.28, 0.18, 0.08, and the sum of the reactions are $R_1 = +99.2$ and $R_2 = +196.6$ kips. Hence the shear on the left of the pivot pier is $+99.2 - 277.0 = -177.8$ kips; this is very near the maximum, a slightly larger value being possible if the pilot wheel be upon the second span.

For Fig. 27 the exact position to give absolute maximum positive moment is uncertain, but a rule sufficiently accurate is to put the third driving wheel at about four-tenths of the span from the left end. This brings two tender wheels on the span; the values of k are 0.04, 0.14, 0.30, 0.40, 0.50, 0.60, 0.76, and the reaction R_1 is found to be 120.2 kips. The positive bending moment under the third driver is then 1538 kip-feet, or 1 538 000 pound-feet, and this is greater than any other for this position of the loads.

For Fig. 28 the exact position of the facing locomotives can be found only by trial. If the pilot wheels be 3 feet from the pivot the last tender wheels are 1 foot from the end supports, and the values of k are 0.02, 0.12, 0.22, 0.32, 0.48, 0.58, 0.68, 0.78, 0.94. The reactions R_1 and R_3 due to these loads are $+115.37$ and -19.25; hence for loads on both spans $R_1 = +96.12$ and the moment at the pier is -1925 kip-feet, or $-1\,925\,000$ pound-feet.

The dead load shears and moments, taking the dead load as 1200 pounds per linear foot per girder, are also computed for the same sections. In the following table both live and dead load

	Shear at end.	Shear at pier.	Moment at 20′ from end.	Moment at Pier.
Dead load	0	− 60 000	− 240 000	− 1 500 000
Live load	+ 149 600	− 177 800	+ 1 538 000	− 1 925 000
Maximum	+ 149 600	− 237 800	+ 1 298 000	− 3 425 000
Web sections	24.9 sq. in.	39.6 sq. in.		
Depth of girder			5.6 feet	8 feet
Flange stresses			231 800	428 100
Flange sections			25.8 sq. in.	47.6 sq. in.

results are placed, and the maximum values found. Taking 6000 pounds per square inch as the unit shearing stress, the web sections result by dividing the shears by this. The girder is to be 4 feet deep at the end, and 8 feet deep over the pier; the flange stresses then result by dividing the moments by the corresponding depths. Taking 9000 pounds per square inch as the allowable stress for tension or compression, this divided into the flange stresses gives the net flange sections required.

Locomotive turntables are usually plate girders; if the ends just touch the supports under dead load the full live load produces stresses according to the laws of continuous girders, and the stresses may be computed by the methods above given.

Prob. 17. Show that the greatest positive moment due to a single load P on Fig. 27 occurs when $k = 0.43$. Also that the greatest negative moment at the center support due to a single load P occurs when $k = 0.58$.

ART. 18. A RIM-BEARING CONTINUOUS TRUSS.

A railroad swing bridge of the Pratt type is continuous over a rim-bearing turntable and has locked ends so that the dead load stresses are the same whether it be opened or closed. The

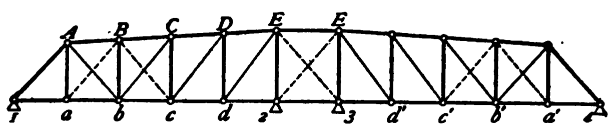

Fig. 29.

spans are 75, 18, and 75 feet, the depth being 17 feet at Aa and 21 feet at the middle. The dead panel load is 18 kips, of which 13.5 kips are on the lower chord, and the live panel load is 54 kips. The diagonals in the central panel are made very heavy, so that full continuity is secured. It is required to compute the maximum and minimum stresses.

As this truss has an inclined upper chord the method of moments will be used for all members. For all web members the centers of moments are at the point where the two chords meet; this is at a distance $16p$ to the left of the left support, p being the panel length of 15 feet. The lever arms for the verticals Aa, Bb, etc., are $17p$, $18p$, etc.; those for the diagonals Ab, Bc, Cd, De are $13.06p$, $14.13p$, $15.20p$, $16.27p$; those for Ba and Cb are $13.50p$ and $14.60p$.

When the bridge is open each arm is a cantilever and the dead load stresses are computed by the method of Part I, Art. 71. When the bridge is closed these dead load stresses are unaltered by the locking bolts, but the live load stresses must be those given by the laws of continuity in the last chapter.

The reactions due to each live panel load of 54 kips are first found by the formulas of Art. 6; these are given in Art. 10. Then the live load is put into the position to give the largest shear for each web member or the largest moment for each chord member; the positions are stated in Arts. 7 and 8. The equations of moments for each member are then stated and solved. For example, for Dc,

$$-77.15 \times 16p + 54(17p + 18p + 19p) - S \times 15.2p = 0,$$
$$-7.03 \times 16p - S' \times 15.2p = 0,$$

whence $+110.6$ and -7.4 kips are the limiting stresses in Dc caused by the live load. In fact all the equations in Art. 10 apply here, if the dead load reactions and panel loads be thrown out.

In the following table the dead and live load stresses for some of the members are given in kips, and the maximum and

LOCKED ENDS.	Cb	Ab	Cc	ab	cd	CD
Dead load	+41.1	0	−35.1	−22.5	−81.0	+48.0
Live load	+61.3 (−22.9)	+53.6 (−21.8)	−45.6	+74.3	+21.1 −11.6	−71.5
Maximum	+102.4	+31.8	−80.7	+74.3	−92.6	+48.0
Minimum	+18.2	0	−35.1	−22.5	−59.9	−23.5

minimum stresses are then found. It is seen that the diagonal *Bc* has no stress, since the member *Cb* is always under tension, while *Ab* is needed to take tension under live load. The lower chords as far as *c* may take either tension or compression, while beyond *c* they are always in compression.

If this bridge have lifted ends, and if the amount of lifting be such that the dead load stresses be those of a purely continuous beam, then there are two cases of dead load to be considered, as in the following tabulation. Here the live load

LIFTED ENDS.	*Cb*	*Ab*	*Cc*	*ab*	*cd*	*CD*
Dead load, open	+41.1	0	−35.1	−22.5	−81.0	+48.0
Dead load, shut	+12.8	+10.6	−4.5	+24.8	+3.2	+23.8
Live load {	+61.3 (−22.9)	+53.6 (−21.8)	−45.6	+74.3	+21.1 −11.6	−71.4
Maximum	+74.1	+64.2	−49.1	+99.1	−81.0	+48.0
Minimum	−10.1	0	−4.5	−22.5	+24.3	−47.6

can only be combined with the second case of dead load to find the maximum and minimum.

Prob. 18. Compute the stresses for *A 1*, *Aa*, and *1 a* for the case of locked ends; also for the case of lifted ends.

ART. 19. PARTIALLY CONTINUOUS SWING BRIDGES.

A partially continuous swing truss is a structure in which the continuity is imperfect, owing to the omission of diagonals in the span or panel over the pier. As this panel has horizontal chords it follows that no shear can be transmitted through it, and hence the continuity is defective. Let the length of the middle span be nl, and that of each of the other spans be l. If a load P be placed in the first span at a distance kl from the support *1*, the shear and moment diagrams will be as shown in Fig. 30, the reactions R_3 and R_4 will be equal with opposite signs, and the moments M_2 and M_3 will be equal. It is seen that while R_4 is negative, R_2 and R_3 are both positive.

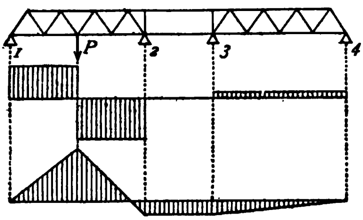

Fig. 30.

This is a favorable condition for rim-bearing trusses, since all tendency to tipping of the turntable is thus avoided. Sometimes diagonals are inserted over the pier, but made very light, so that they do not really carry shear. In fact all swing bridges of large span are now arranged so that they are partially continuous in respect to the transmission of shears.

The theorem of three moments is inapplicable to this case because the elastic curve is not continuous over the supports *2* and *3*, it having in fact a cusp at each of these points on account of the break in shears. Some other principle must hence be used to determine the reactions due to load P. The principle of least work, explained in Art. 83 of Part I, will here be employed; this asserts that the reactions must be such as to render the work of the internal stresses a minimum. Now regarding the truss as a beam of constant cross-section, the work of the internal stresses is proportional to the sum of the squares of all the bending moments (see Mechanics of Materials, Art. 109). For a section between the left end and the load the bending moment is R_1x, for a section on the right of the load it is $R_1x - P(x - kl)$, for the middle span it is R_4l, and for the right-hand span it is R_4x. The sum of the squares of all the bending moments hence is,

$$\int_0^{kl} R_1^2x^2dx + \int_{kl}^{l}(R_1x - Px + Pkl)^2dx + \int_0^{nl} R_4^2l^2dx + \int_0^{l} R_4^2x^2dx,$$

and the values for R_1 and R_4 must be such as to make this a minimum.

The reactions in Fig. 30 are subject to the two static conditions of equilibrium: first, that the sum of the reactions equals the load; second, that the sum of the moments of the reactions equals the moment of the load, or,

$$R_1 + R_2 + R_3 + R_4 = P, \quad R_4(2l + nl) + R_3(l + nl) + R_2 l = Pkl.$$

Also, the condition of partial continuity is that no shear exists in the middle span, or $R_1 + R_2 - P = 0$. From these three equations are found,

$$R_2 = P - R_1, \quad R_4 = -R_3 = R_1 - P(1 - k),$$

and hence R_2, R_3, R_4, are known as soon as R_1 has been found. To determine R_1 let the integrations in the above expression be performed and then R_4 be replaced by its value in terms of R_1. An expression is thus found containing only R_1, P, k, and l, and making this a minimum with respect to R_1 the value of R_1 results. The reactions then are,

$$R_1 = P(1 - k) - \frac{P}{4 + 6n}(k - k^3),$$

$$R_2 = Pk + \frac{P}{4 + 6n}(k - k^3),$$

$$R_3 = -R_4 = \frac{P}{4 + 6n}(k - k^3).$$

Here if $n = 0$, the reactions reduce to the values found in Art. 5 for the case of two equal spans. Another and perhaps better method of deducing these formulas is given at the end of Art. 22.

A continuous truss of three spans has R_3 negative and R_4 positive for a load in the first span, but the reverse is the case for a partially continuous truss like Fig. 30. The distribution of live load to give the largest shears is hence different from the cases shown in Figs. 11 and 12 of Art. 7, and is similar to those of Figs. 8 and 9. It is plain on reflection that the live load distribution to give positive shear for a section in the first span is as shown in Fig. 31, and that the distribution to give negative shear is as shown in Fig. 32.

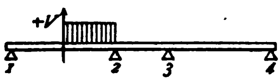

Fig. 31.

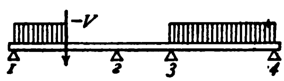

Fig. 32.

Reasoning by the method of Art. 8, the following distributions of live load are deduced for the moments. For the first span there is a critical point i whose distance from the left support is,

$$i = \frac{4+6n}{5+6n} l.$$

For all sections on the left of this point the largest positive moment occurs when the first span is fully loaded, and the largest negative moment when the last span is fully loaded with the live load. For a section on the right of the point i the

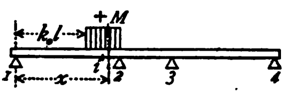

Fig. 33.

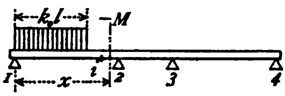

Fig. 34.

largest positive moment occurs when the live load is placed as in Fig. 33, and the largest negative moment when it is placed as in Fig. 34. Here the distance $k_0 l$ is found from

$$k_0^2 = 5 + 6n - (4+6n)\frac{l}{x},$$

which gives the load limits for any value of x greater than i. For instance, if $n = 0.25$, then $i = \frac{22}{26} l$; and for $x = \frac{23}{26} l$, the value of k_0 is 0.536. When $x = l$, then $k_0 = 1$; that is, both spans are fully loaded to give largest negative moment in the middle span.

The above formulas for the reactions were deduced by MER-

RIMAN in 1895, and published in the Railroad Gazette of Sept. 6, and in Engineering News of Sept. 5 for that year. Being derived under the assumption that the truss is a beam of uniform cross-section, they are not strictly correct, but they have the same validity as all other formulas for reactions given in the preceding pages. In Art. 22 it will be shown how accurate values for the reactions of a trussed structure may be computed.

Prob. 19. When the partially continuous truss in Fig. 30 has its first span fully loaded, show that the reactions are,

$$R_1 = \frac{7+12n}{16+24n} wl, \quad R_2 = \frac{9+12n}{16+24n} wl, \quad R_3 = -R_4 = -\frac{wl}{16+24n}.$$

Also find the reactions when both spans are fully loaded.

ART. 20. A PARTIALLY CONTINUOUS TRUSS.

In Fig. 35 is shown the modern type of truss for a partially continuous swing bridge. The members drawn in light lines take only tension, those in heavy lines may take either tension or compression. The truss is partly of the Pratt and partly of the Baltimore type, this being the arrangement which has

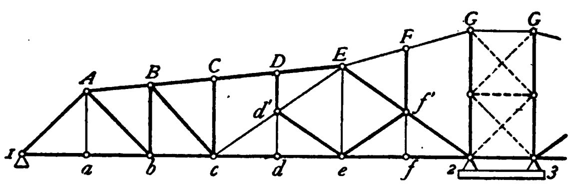

Fig. 35.

been found to be most conducive to economy of material (see Part I, Art. 89, and Part III, Art. 3). The broken members in the tower serve only to stiffen it and carry no shear, hence the truss is only partially continuous. Let the panel length be 20 feet, each side span being thus 140 feet, and let the middle span be also 20 feet. Let the depths Aa, Ee, $G2$, be 20, 28, 40 feet respectively. Let the dead panel load be 24 000 pounds or 24 kips, one-fourth of this being on the upper chord, and the live panel load 68 kips, all on the lower chord. The ends are to be locked, so that the dead load stresses are the same whether the bridge be open or closed.

The stresses due to dead load are readily computed arithmetically by the principles of Part I, or graphically by the methods of Part II. The method used in Part I, Art. 71, is in particular an advantageous one for the webbing. The dead panel load at *1* is 9 tons, at *a*, *b*, etc., 18 tons, and at *A*, *B*, etc., 6 tons.

From the formulas of the last article, the reactions due to the live panel load of 68 kips are computed, *n* being 1/7:

Load at	*a*	*b*	*c*	*d*	*e*	*f*	Sum
Reaction R_1=	+56.33	+44.90	+33.96	+23.77	+14.52	+6.52	+180.00
Reaction R_4=	−1.96	−3.67	−4.90	−5.37	−4.91	−3.19	−24.00

When the first span is fully loaded, the reaction R_1 is +180.00, when the right-hand span is loaded it is −24.00, and when both spans are loaded it is +156.00 kips.

It is customary in many bridge offices to determine the chord stresses due to live load by computing them for two cases;

CHORDS.	*AB*	*BE*	*EG*	*GG*	*1b*	*bc*	*c2*
Dead load	+38.4	+82.9	+296.0	+283.5	−9.0	−38.2	−186.4
Live load on *1–2*	−266.8	−281.4	+87.7	+84.0	+180.0	+265.5	+205.7
Live load on *3–4*	+43.9	+60.3	+87.7	+84.0	−24.0	−43.6	−85.7
Maximum	−228	−199	+471	+452	+171	+227	−272
Minimum	+82	+143	+296	+283	−33	−82	+19

first, when the left span is loaded; and second, when the right span is loaded. These are then combined with the dead load stresses to find the maxima and minima. This is correct for all chords except those whose center of moments lies beyond the critical point *i* (Figs. 28 and 29). For the truss in hand

$i = \frac{34}{41} l$, and the method is strictly correct for all chords, since the distance of the center E from the left support is less than this value of i. The above table gives the chord stresses for dead load and for these two cases of live load, as also the maxima and minima resulting from their combination.

The vertical members Aa, $d'd$, $f'f$ are stressed only by the lower panel load, the limiting values being $+86$ and $+18$ kips. The verticals Cc, Dd', Ff' have the stress -6 kips due to the upper panel load. The vertical Ee has $+136$ and $+42$ kips. The auxiliary braces $d'c$ and $f'c$ are stressed only by the panel loads at d' and f', the limits being -80 and -21 kips. The end post $A1$ has $+12.7$ under dead load, and -254.8 and $+33.1$ for the above cases of live load, which give -242 and $+46$ for the maximum and minimum. The center post G_2 has the maximum -135 and the minimum -85 kips.

For the remaining web members it is necessary to put the live load in the position to give the largest positive and negative live load shears for each, as shown in Figs. 31 and 32, and then to compute the corresponding stresses. These are given in the following table, and by combining them with the dead load

WEB.	*Ab*	*Bb*	*Bc*	*d'c*	*Ed'*	*f'2*	*Ef'*
Dead load	−41.3	+47.2	−69.2	+126.9	+147.8	−118.5	−97.6
Live load {	+143.1	−64.5	+83.9	−58.6	−57.2	+2.1	+2.1
	−50.4	+68.5	−95.0	+180.7	+240.0	−353.8	−294.5
Maximum	+102	+116	−164	+307	+388	−472	−392
Minimum	−92	−17	+15	+68	+91	−116	−95

stresses, the final values are found. These are in kips, one kip being 1000 pounds.

It is seen that the stresses found for this truss are such as to enable a large part of each chord to be made of uniform section, and that the same is also true for a number of pieces of the webbing. This is one of the reasons that render it a highly economic type of truss.

Prob. 20. If the webbing be computed for the live loads used for the chords, show that the maximum stress found for *Bc* will be only 56 per cent of its true value.

ART. 21. DEFLECTION OF A SWING TRUSS.

The deflection of a swing truss due to dead load is readily computed by the method explained in Part I, Art. 81. Without repeating the reasoning there given, the method will be now briefly stated. Let S_1, S_2, S_3, etc., be the stresses in the members due to the dead load when the bridge is open. Let T_1, T_2, T_3, etc., be the stresses due to a load Q at the free end of the truss. Let L_1, L_2, L_3, etc., be the lengths of the members, and A_1, A_2, A_3, etc., the areas of their cross-sections. The deflection at the end is then given by

$$\Delta = \frac{1}{QE}\left(\frac{S_1T_1L_1}{A_1} + \frac{S_2T_2L_2}{A_2} + \frac{S_3T_3L_3}{A_3} + \dots\right) = \frac{1}{QE}\Sigma\frac{STL}{A}, \quad (1)$$

in which the coefficient of elasticity E is regarded as having the same value for all the members of the truss. In respect to the load Q it is to be noted that its value may be taken as anything convenient, 1000 pounds, or one kip, being frequently used. As each stress T is proportional to Q, the latter really occurs both in the numerator and denominator of the expression, and hence its actual value is unimportant.

In the case of a truss which is built, the actual values of the areas of the cross-sections are to be determined by measurement in the field or from the working drawings. In the case of a truss which is under design, the cross-sections must be determined before the deflection can be computed. An example showing the method of procedure will now be given.

Take the truss of the last article and suppose that it be required by the specifications that the working unit stresses shall be as follows for finding the net sections: For tensile members $7000\left(1 + \frac{\text{min}}{\text{max}}\right)$ when there is no reversal of stress, and $7000\left(1 - \frac{1}{2}\frac{\text{min}}{\text{max}}\right)$ when there is reversal; for compression members $5000\left(1 + \frac{\text{min}}{\text{max}}\right)$ when there is no reversal of stress, and $5000\left(1 - \frac{1}{2}\frac{\text{min}}{\text{max}}\right)$ where there is reversal, these being in pounds per square inch, and both max. and min. being taken in the formulas without regard to sign. In the following table the first column designates all the members of the truss in Fig. 35, which receive stress under a load at the end *1*, these being

1 MEMBER.	2 MAX. STRESS.	3 MIN. STRESS.	4 MIN./MAX.	5 UNIT STRESS.	6 AREA. A. sq. in.	7 LENGTH. L. feet.	8 S	9 T	10 $\frac{STL}{A}$
AB	−228	+82	0.36	4.1	56	20.1	+38.4	+43.9	605
BE	−199	+143	0.71	3.2	62	60.3	+82.9	+60.3	4862
EG	+471	+296	0.63	11.4	42	41.8	+296.0	+87.7	25836
GG	+452	+283	0.63	11.4	40	10.0	+283.5	+84.0	5954
$1b$	+171	−33	0.19	6.3	27	40.0	−9.0	−24.0	320
bc	+227	−82	0.36	5.7	40	20.0	−38.2	−43.6	833
$c2$	−272	+19	0.07	4.8	57	80.0	−186.4	−85.7	22420
23	−452	−283	0.63	8.1	56	100	−283.5	−84.0	4252
$A1$	−242	+33	0.14	4.6	53	28.3	+12.7	+33.1	225
Ab	+102	−92	0.90	3.9	26	28.3	−41.3	−29.8	1340
Bb	+116	−17	0.15	6.4	18	22.0	+47.2	+21.1	1217
Bc	−164	+15	0.09	4.8	34	29.7	−69.2	−23.2	1402
Ed'	+388	+91	0.23	8.6	45	24.4	+147.8	+31.4	2516
$d'c$	+307	+68	0.22	8.6	36	24.4	+126.9	+31.4	2701
Ef'	−392	−95	0.24	8.7	44	24.4	−97.6	+2.1	−114
$f'2$	−472	−116	0.25	8.7	54	24.4	−118.5	+2.1	−112
$G2$	−135	−85	0.63	8.1	30	40.0	−85.0	−25.2	2856

$\Sigma STL/A = 77113$

all that it is necessary to consider to compute the deflection of that end. The second and third columns contain the maximum and minimum stresses in kips computed in the last article. The fourth column gives the ratio of the minimum to the maximum stress, regardless of sign. In the fifth column are the working unit stresses as computed from the specified rules above stated; these are in thousands of pounds per square inch; that is, in kips per square inch. These unit stresses divided into the maximum stresses give the preliminary net areas of the members in square inches, and these are arranged in the sixth column. The lengths of the members in feet are in the seventh column; for *GG* and *23* only one-half their lengths are stated because it is intended to include only one-half of the truss. The eighth column gives the dead load stresses when the bridge is open. In the ninth column are given the stresses due to a load of 24 kips at the end of the truss; *Q* is here taken as 24 kips because the chord stresses due to that load have been already obtained in the last article, and the web stresses can be derived by a little additional computation. The last column contains the values of STL/A for the different members. The sum of these values is 77 113; since all lengths have been taken in feet, this is to be multiplied by 12 to express it in inches. Now this truss being steel, E is 30 000 000 pounds per square inch, or 30 000 kips per square inch. Then from the above formula

$$\Delta = \frac{77\,113 \times 12}{24 \times 30\,000} = 1.29 \text{ inches},$$

which is the deflection of the truss due to the dead load. The end of the lower chord at *1* is hence lower than the lower chord at *2* by this amount, and the levels of the bridge seats must be arranged accordingly.

The values of A used in the above example are preliminary net areas, no allowance having been made for rivets, or for stresses due to flexure and to length of compression members, all of which must be taken into account in an actual case. The section used for the center post $G2$ has been taken larger than the net value as it is to support an engine house, and is also affected by wind. In all final computations for deflection the gross areas of the members must be used.

The above formula for the deflection is a general one and may be used to determine the deflection at any point of the truss due to a live load in any given position. In this case S denotes the stresses due to the given live load, and T the stresses due to any load Q placed at the point whose deflection is desired; the value of Q is generally taken as 1000 pounds, or one kip, unless the stresses due to some other load at that point have already been computed. Here the summation must be extended to include both arms of the truss, since the stresses are different in the two halves. If S and T are of different sign it is to be noted that their product is negative, but this will not usually occur for many members.

If it be desired to determine the deflection at a certain point due to a load Q at that point, the stresses S and T are the same, and the formula becomes

$$\Delta = \frac{1}{QE} \Sigma \frac{T^2L}{A}. \qquad (2)$$

If the load Q be taken at the end of the truss this formula gives the deflection of the end due to that load. A load P at the end evidently produces P/Q times this deflection.

Prob. 21. Compute the amount of the above deflection of 1.29 inches which is due to the actual panel load of 9 kips at the end.

ART 22. TRUE REACTIONS FOR SWING TRUSSES.

In the preceding pages stresses have been computed for swing bridges by the use of formulas for reactions which, although accurate for beams of uniform cross-sections, do not strictly apply to trusses. In a swing truss the chords are usually not parallel and they are not of uniform section throughout, while such requirements are implied in deducing the formulas for reactions. Moreover, these formulas are derived from the bending moments alone, while in reality the influence of the webbing is considerable. It is now to be shown how, after a preliminary design of the truss has been made, more accurate values of the reactions may be determined. With these new reactions the stresses may be recomputed, and the sections revised. Then, if necessary, reactions more accurate still may be derived.

The stresses are first to be computed by the method of the preceding articles, using the common formulas for reactions. Then from these stresses, using the unit stresses assigned in the specifications, the cross-sections of the members are to be derived. Let these cross-sections be called A_1, A_2, A_3, etc. Then let it be required to determine the reaction R_1 due to a load P at any position on the first span.

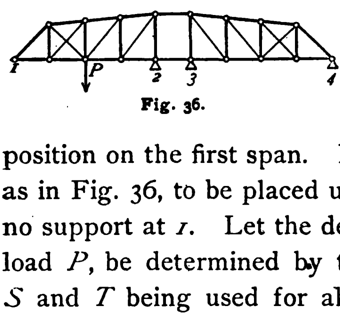

Fig. 36.

For this purpose suppose the truss, as in Fig. 36, to be placed upon the supports *2*, *3*, *4*, there being no support at *1*. Let the deflection Δ of the end *1*, due to this load P, be determined by the method of Art. 21, the stresses S and T being used for all the members of the truss. Also let the deflection Δ' of the end *1* due to a load Q at that end be determined. The formulas for these deflections are:

$$\Delta = \frac{1}{QE} \Sigma \frac{STL}{A}, \qquad \Delta' = \frac{1}{QE} \Sigma \frac{T^2 L}{A}.$$

Now, under the above supposition that there is no support at *1*, these deflections exist; but if the end be raised the amount Δ a reaction R_1 due to P results, and if it be raised the amount Δ' the reaction Q results. These reactions are proportional to the corresponding deflections; hence,

$$R_1 = Q\frac{\Sigma STL/A}{\Sigma T^2L/A}, \qquad (1)$$

which is a general formula for the left reaction due to a load P at any point, S representing the stresses due to P when the left end is free and the right end is locked, and T representing the stresses due to a load Q at the end. Here Q may have any value as it is a factor in every T. The summation must be extended to include all the members of the truss in all spans.

The above formula may be modified so that it will be unnecessary to compute stresses for the right-hand part of the truss. Referring to Fig. 36 let kl be the distance of P from the left end; then as there is no shear in the middle panel, the reaction R_2 is equal to P, and the reactions R_3 and R_4 are $P(1-k)$, the latter being negative. Accordingly the stresses caused by P in the right-hand part of the truss are those due to a load $P(1-k)$ at *4*. In like manner the load Q at *1* produces a negative reaction Q at *4*, and the stresses due to Q are the same for both parts of the truss. Hence, the formula (1) may be written

$$R_1 = \frac{Q\Sigma_1\frac{STL}{A} + P(1-k)\Sigma_1\frac{T^2L}{A}}{2\,\Sigma_1\frac{T^2L}{A}},$$

and this reduces to the simpler form,

$$R_1 = \tfrac{1}{2}P(1-k) + \tfrac{1}{2}Q\frac{\Sigma_1 STL/A}{\Sigma_1 T^2L/A}, \qquad (2)$$

in which Σ_1 denotes that the summation covers all members between the left end of the truss and the middle of the middle span.

To illustrate the method take the truss of Fig. 37 whose stresses were computed in Art. 20, and for which the approximate sections were determined in Art. 21. Using these sections,

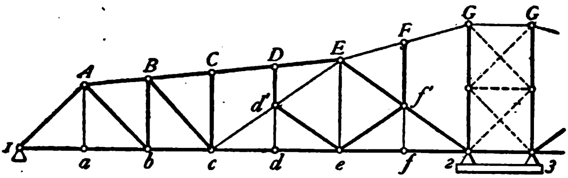

Fig. 37.

let it be required to compute the true reaction R_1 due to a load of 68 kips at the panel point *c*. The load Q will be taken as 24 kips, because the stresses T due to this load were computed in Art. 20. The values of T, L, and A for all members between *1* and the middle of the span *2–3* are given in the table of Art. 21. Squaring each T, forming the quantities T^2L/A and adding, there is found $\Sigma_1 T^2L/A = 31638$. The stresses S, due to a load of 68 kips at *c* when there is no support at *1*, are then computed; namely, $EG = +142.0$, $GG = +136.0$, $c2 = -97.2$, $2\text{–}3 = -136.0$, $Ed' = d'c = +118.6$, $Ef' = f'2 = -47.4$, $G2 = -40.8$ kips. Multiplying each of these by its T and L, and dividing by A, there is found $\Sigma_1 STL/A = 34780$. Then from formula (2) the reaction at the left end due to this load is,

$$R_1 = \frac{68 \times 4}{2 \times 7} + \frac{24 \times 3478}{2 \times 31638} = 32.62 \text{ kips},$$

while the value computed from the beam formula and used in Art. 20 was 33.96 kips.

In like manner the true reactions due to all the loads for the truss are found, and the following is a comparison of their values with those used in the previous computations:

Load at	*a*	*b*	*c*	*d*	*e*	*f*	Sum
R_1(Art. 20) =	56.33	44.90	33.96	23.77	14.52	6.52	180.00
True R_1 =	54.84	43.95	32.62	21.32	13.16	5.45	171.34

These are the true reactions for the truss when built with the sections given in the sixth column of the table in Art. 21. By the use of these true reactions the stresses may be recomputed and the sections revised. The new maximum stresses will be about 7 per cent smaller than the old ones for the chords near the left end, and about 12 per cent larger for the chords near the middle span.

The formula (2) deduced above is a general one, good for all swing trusses of two or three spans, and it may be used for computing the reactions in the first instance instead of using the beam formulas of the preceding articles. In doing this the cross-sections are unknown, and they may be taken as equal, or A be made unity; further, the summation may be confined to the chord members only. For example, if this be done for the load of 68 kips at c, there is found $R_1 = 31.83$, a result which agrees very well with the second value.

Formula (2) may also be applied to a beam of constant cross-section by replacing S and T by their values in terms of bending moments, and changing the sign of summation to that of integration. By the methods of Mechanics of Materials, the expresssion $\Sigma \frac{STL}{AE}$ becomes for a solid beam $\int \frac{Mmdx}{EI}$, in which M is the moment due to the load, and m the bending moment due to a load unity at the point whose deflection is desired (see also Art. 76). Accordingly, (2) reduces to

$$R_1 = \tfrac{1}{2} P(1 - k) + \tfrac{1}{2} \frac{\int_{kl}^{l} Mmdx + \int^{\frac{1}{2}nl} Mmdx}{\int_0^{l} m^2dx + \int_0^{\frac{1}{2}nl} m^2dx},$$

in which the moments are to be taken as if there were no support at the left end. Thus, for the case of the partially continuous truss of Fig. 36, the values of M and m for the first span are $-P(x - kl)$ and $-x$, and while for the middle span they are $-P(l - kl)$ and $-l$. Inserting these and performing the integrations, there results,

$$R_1 = P(1 - k) - \frac{P}{4 + 6n}(k - k^3),$$

which is the same as deduced in Art. 19 by another method.

Prob. 22. Compute the live load stresses for AB and GG of Fig. 37, using the true live load reactions there given. Then find the revised maximum and minimum stresses for these members.

ART. 23. DOUBLE SWING BRIDGES.

In Fig. 38 are represented two swing trusses, each of which is partially continuous by virtue of the omission of diagonals over the pier, and by the break in the chords at M. The river here occupies the space CD, the two piers BC and DE being built at the banks. Each bridge swings upon its own pier, and when both are parallel to the banks the entire river is free for the water traffic. When the bridge is closed the two river arms are locked at M, and the land arms are locked at A and F. If the locking bolts bring no upward pressures at the ends, the dead load stresses are the same when the bridge is closed as when it is open. The live load stresses, however, are governed by the laws of partial continuity, and these can be computed as soon as the reactions are determined.

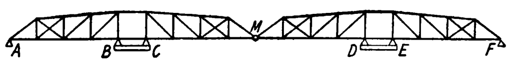

Fig. 38.

CASE I. — Let the spans BC and DE have the length nl, and the other spans the length l. Let a single load P be upon the truss at a distance kl from the support A, and let the reactions at A, B, C, D, E, F be denoted by $R_1, R_2, R_3, R_4, R_5, R_6$; to determine these, six conditions are required. The principles of statics furnish two conditions, namely, that the sum of the vertical forces equals zero, and that the sum of the moments of these forces equals zero, or, algebraically,

$$R_1 + R_2 + R_3 + R_4 + R_5 + R_6 - P = 0,$$
$$-Pkl + R_2 l + R_3(1+n)l + R_4(3+n)l + R_5(3+2n)l + R_6(4+2n)l = 0.$$

This truss is partially continuous in three respects; there is no shear in the span BC, there is no shear in the span DE, and there can be no moment at M. These conditions furnish the three equations,

$$R_1 + R_2 - P = 0, \qquad R_6 + R_5 = 0,$$
$$R_4 l + R_5(1+n)l + R_6(2+n)l = 0,$$

and thus five equations are established between the six unknown reactions. From these are deduced,

$$R_2 = P - R_1, \qquad R_3 = -R_4 = -R_5 = R_6 = P(1-k) - R_1,$$

and thus all reactions are known when R_1 has been found.

The value of R_1 may be derived by the principle of least work, as was done by MERRIMAN in 1895, or it can be deduced by the application of the general formula (1) of Art. 22. Without giving the algebraic work, the result may be stated, namely,

$$R_1 = P(1-k) - \frac{P}{8+12n}(k-k^3),$$

from which numerical values may be computed either directly or by the help of the table in Art. 13.

CASE II. — Let a single load P be on the span CM at a distance kl from the middle joint M. Then by a similar method are found,

$$R_2 = -R_1, \quad R_3 = Pk - R_1, \quad R_4 = R_5 = -R_6 = P(1-k) + R_1,$$

and then the formula for R_1 is,

$$R_1 = -\frac{1}{2}P(1-k) - \frac{P}{8+12n}(k-k^3).$$

These formulas may also, by a change of subscripts, be applied to loads on the span *MC*, and those of Case I be applied to loads on the span *EF*. Also, they may be applied to a uniform load over the entire span by replacing *P* by *wd*(*kl*), and integrating between the limits 0 and 1.

Few draw bridges of this kind have been built. The most important one is that at Cleveland, Ohio, by W. P. RICE in 1895. The total length is 279 feet, each river arm being 65 feet, while the short spans over the piers are 15 feet. It is a highway bridge with the floor on a three per cent grade. For description and illustrations, see Engineering News, August 8, 1895.

When the spans *AB* and *CM* in Fig. 38 are of different lengths, let $AB = EF = l$, $BC = DE = nl$, and $CM = MD = n'l$. The following formulas will then furnish the reactions:

CASE I. — A load on the span *AB* at a distance *kl* from the left end. The reaction at *A* is

$$R_1 = P(1 - k) - \frac{P}{m}(k - k^3),$$

in which $m = 4 + 4n' + 12n$. The other reactions are,

$$R_2 = P - R_1, \quad R_6 = -R_5 = -n'R_4 = n'R_3 = P(1 - k) - R_1.$$

CASE II. — A load on the span *CM* at a distance $k(n'l)$ from the middle joint *M*. The reaction at *A* is,

$$R_1 = -\tfrac{1}{2}Pn'(1 - k) - \frac{n'^2}{m}P(k - k^3),$$

and also $R_2 = -R_1$. The others are given by

$$n'R_3 = n'Pk - R_1, \quad R_4 = n'R_5 = -R_6 = Pn'(1 - k) + R_1.$$

In both these cases, if n' be made unity, the formulas reduce to those previously given. Also, if n be made zero, they apply to the case of center bearing pivots, as in Fig. 39, where the sum of R_2 and R_3 gives the reaction at B, and the sum of R_4 and R_5 gives the reaction at C.

Prob. 23. For the case of Fig. 38, let the main spans be each 40 feet long, and the pivot spans be each 30 feet long. Compute the reactions due to a load of 100 pounds at the middle of the span AB. Also compute the reactions due to a load of 100 pounds at the middle of the span CM.

ART. 24. SIMPLE SWING BRIDGES.

In Fig. 39 is shown a draw bridge which is formed by two simple spans. The central turntable supports the two water

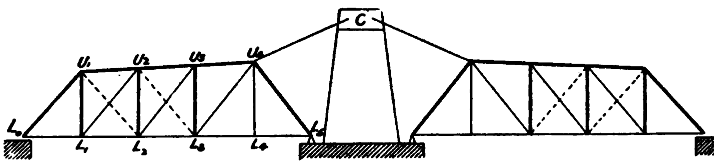

Fig. 39.

ends of these spans, and also a tower having an engine at C. When the bridge is to be swung open tension is brought by the engine upon the two members CU_4 so as to lift the land ends of the bridge from the abutments. When thus lifted each span is a cantilever arm, and the entire structure is revolved on the turntable until it is at right angles with its fixed position. When revolved back into place the tension in CU_4 is relaxed, and each span becomes again a simple truss.

The methods of computing the stresses for this case need not here be explained, as they have been already given in Part I, Art. 71. It is seen that when the bridge is open the upper chord is in tension and the lower in compression, and that when it is closed the reverse is the case. The chords must be proportioned to resist both tension and compression, and they are slightly heavier than those of the partially continuous truss of Fig. 37.

This type of draw bridge is increasing in favor, since by its use all the uncertainties of continuity are avoided. The first important one erected was in 1889 on the Baltimore and Ohio Railroad over the Arthur Kill on Staten Island, N.Y., which has a total length of 496 feet; for description see Railroad Gazette, June 22, 1888.

Prob. 24. Ascertain, by consulting the engineering journals, what type of truss is used in the following swing bridges, and give a sketch of each with the principal dimensions, the loads used, and the character of the end and center arrangements: (1) The bridge over the Missouri river at Omaha; (2) the bridge over the Thames river at New London, Conn; (3) the bridge over the Arthur Kill on Staten Island, N.Y.; (4) the bridge over the Harlem river at New York; (5) the bridge built in 1897 at Duluth, Minn.; (6) the four track bridge built in 1895 over the Bronx river at New York.

ART. 25. HORIZONTAL ROLLING DRAW BRIDGES.

A draw bridge which is pushed out horizontally on rollers over the river and drawn back upon one bank when the waterway is to be cleared, is plainly applicable only to a short span. When the bridge is closed it consists of two continuous spans with respect to live load, while the dead load acts in the same manner as if the bridge were open, provided that the ends are merely locked. The stress computations are hence similar to those given in Art. 16, and need no further explanation.

Figure 40 shows a more common case, where there are two parts AM and MD, which are locked together at M when the bridge is closed. When the bridge is to be opened AM is rolled back upon the left bank and MD upon the right bank. Here the river is underneath BC, and the pivots at B and C, together with the spans AB and CD are upon wheels or other arrangements whereby they can be rolled horizontally backward from the shores. The shore spans AB and CD may be shorter in length than the river arms, and may have different styles of webbing. The dead load stresses are here the same whether the bridge be open or closed, as the locking pin at M can bring no reactions upon the truss.

Fig. 40.

The live load stresses are computed by the methods of the previous articles as soon as the reactions due to a single load are known. Let $AB = CD = l$, and $BM = MC = n'l$ where n' is usually greater than unity. Let the reactions at A, B, C, D, be called R_1, R_2, R_3, R_4.

CASE I. — A load P on the span AB at a distance kl from A. The reactions are given by the formulas,

$$R_1 = P(1 - k) - \frac{P}{4sn'}(k - k^3),$$

$$R_4 = P(1 - k) - R_1, \quad R_3 = -sR_4, \quad R_2 = Pk - R_3,$$

in which, for abbreviation, s represents the number $(1 + n')/n'$.

CASE II. — A load P on the span BM at a distance $k(n'l)$ from the joint M. The reactions are:

$$R_1 = -\tfrac{1}{2} Pn'(1 - k) - \frac{n'P}{4s}(k - k^3),$$
$$R_2 = Pk - sR_1,$$
$$R_3 = -sR_4 = n'sP(1 - k) + sR_1.$$

Here if all spans are equal, $n' = 1$ and $s = 2$.

For example, if $AB = 30$ feet and $BM = 60$ feet, $n' = 2$, and $s = 1\tfrac{1}{2}$. If a load be on the span BM at a distance of 45 feet from M the value of k is 0.75, and from the table in Art. 13 the value of $k - k^3$ is 0.3281. Then from the above formulas, $R_1 = -0.3594\,P$, $R_2 = 1.2891\,P$, $R_3 = +0.2109\,P$, and

$R_4 = -0.1406\,P$. The stresses due to P are now readily computed in all parts of the truss.

Prob. 25. Draw the shear and moment diagrams for the case of a single load on the span AB; also for a single load on the arm BM.

ART. 26. HINGED LIFT BRIDGES.

In Fig. 41 is seen a simple hinged lift draw bridge supported at one end and hinged at the other, it being movable in a vertical plane around the hinge. A cable attached near the free end of the bridge passes over a pulley on the vertical tower and supports a counterweight. An engine in the tower opens and closes the bridge, the office of the counterweight being to economize power.

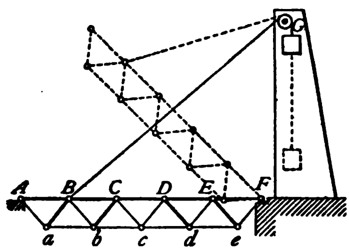

Fig. 41.

When the bridge is closed for the passage of traffic, its trusses are simple ones, which are computed by the methods of Part I. As the bridge begins to open, additional stresses are applied to the truss through the tension in the cable BG. To determine these additional stresses, it will be convenient to resolve the force acting at B into its horizontal and vertical components. The horizontal component causes compression in the chord BF, while the vertical component causes stresses in the webbing. For example, let the truss in the figure have a span of 50 feet and a depth of 6 feet, and let BG be inclined at an angle of 45 degrees. Let the dead load per linear foot per truss be 400 pounds, all taken on the upper chord. The total load W is then 20 000 pounds, and the stress in the cable is 17 680 pounds. This gives a horizontal component of 12 500 pounds, which produces compression in BF, and a vertical component of

12 500 pounds, which causes positive shear in all diagonals between *B* and *F*. As the bridge rises, the stresses due to the cable decrease, and when it reaches a vertical position they become zero; the dead load of the bridge, however, now causes compression in *BF*.

Another method which has been proposed for a hinged lift bridge is shown in Fig. 42. Here there is no tower and no counterweight, but the cable is attached to the truss at *B*, and passes over small pulleys on trestles at *D* and *E* and then to the engine. This method is best adapted to a through bridge, and while the power required to lift the structure is greater than in the previous case, the chord compression due to the pull on the cable is largely avoided. When the bridge is closed, the stresses are exactly as in a simple truss; when it just begins to open, the stresses are as in a cantilever arm. The principles of Parts I and II are entirely sufficient for the complete analysis.

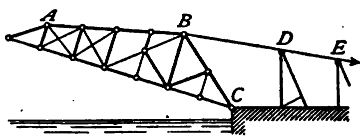

Fig. 42.

Structures of the kind described in this article are sometimes called bascule bridges, particularly in Europe, each arm being termed a bascule leaf. They are generally used for short spans, and are frequently made of plate girders, with a solid floor construction. The largest bascules are those of the Tower bridge in London, built in 1894, where each leaf is 100 feet in length; this has also an overhead bridge, connecting the tops of towers, which may be used by foot passengers when the bascule leaves are open.

Prob. 26. When the truss in Fig. 41 just begins to open, show that the stresses in *Cc*, *Dc*, *CD*, *bc*, due to the dead load and cable pull, are + 3.3, − 3.3, − 20.8, + 12.4 kips. Also find these stresses when the angle *GBF* is 30 degrees.

ART. 27. OTHER FORMS OF LIFT BRIDGES.

A common form of lift bridge for short spans and highway traffic is a simple truss which is lifted bodily by means of cables attached to its ends and passing upwards over pulleys in two towers. A counterweight is usually provided in each tower to balance the weight of the bridge and thus lessen the power required to move it. These have been used for many years over the Erie canal, and in the past decade several large structures have been elsewhere erected. The largest, built by WADDELL in 1893 over the Chicago river, is a highway bridge of 130 feet span and 50 feet width; it has a lift of 141 feet. For description see Railroad Gazette, February 24, 1893, and Transactions American Society of Civil Engineers, January, 1895.

The Scherzer rolling lift bridge, of which two have been built across the Chicago river, consists of two parts or arms which are locked together when the bridge is closed. Each part is supported on pedestals, at the top of which are hinges. When the bridge is to be opened each arm rises around these hinges while the lower portion of the truss rolls upon the abutments. For a fuller description see Journal of Association of Engineering Societies, December, 1895.

Two forms of bascule bridges have been built in Milwaukee, over canals, which have a combined lifting and rolling motion. As the river end rises the shore end falls and the latter is constrained to follow a certain curve by means of guides, these guides being so arranged that the center of gravity of the leaf or arm moves in a horizontal line. Hence no power is expended in lifting weight, but only sufficient is required to overcome inertia and frictional resistance. Descriptions of these structures will be found in Engineering News, March 7, 1895, and April 22, 1897.

Other modifications have also been suggested and tried for lift bridges of short span. One of these is to make the bascule arm in two parts, and as it rises the middle part folds downward upon the other one; this arrangement, which is suitable only for a very light structure, is called a "jack-knife bridge"; see Engineering News, May 23, 1891, and November 27, 1896. In Holland and England there are numerous bascules of the plate girder type, which are generally operated by hydraulic power, whereas in the United States electric power is used for light structures and steam power for heavy ones. The bascule type has the advantage that it can be opened more quickly than the swing bridge and it certainly offers less obstruction to the water way. On the other hand it cannot be economically built for long spans. In any particular case the local conditions must be carefully studied by the engineer, and such a structure be selected that the cost of construction and maintenance shall be a minimum and the local requirements as to land and water traffic be fulfilled in the most advantageous manner.

ART. 28. POWER TO OPEN A SWING BRIDGE.

The power used in opening a swing bridge is expended in two ways: first, in overcoming the inertia of the structure or putting it into motion; and second, in overcoming the frictional resistance of the air and of the pivots, wheels, or other moving parts. The latter class of resistances requires experiment for their complete investigation, while the former depends only upon theory. It is, however, a fact that the laws of resistance of inertia are not commonly well understood, and hence it may be worth the while to attempt here a brief explanation.

Let AA represent the plan of a closed swing bridge which is to be swung horizontally around into the position CC in t seconds. Starting from rest it is to acquire velocity uniformly until it reaches the position BB, where the velocity is a maximum; then the velocity is to uniformly decrease until it comes to rest in the position CC. Let the weight of the bridge per linear foot be w and the length of each span be l, so that the total weight is $2\,wl$.

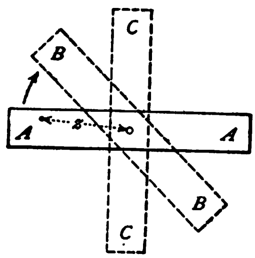

Fig. 43.

At any distance z from the center suppose there to be a weight W which is to be moved horizontally by a constant force F applied at the same distance from the center. Let v be the velocity which this force F gives to the weight W in one second. Let g be the velocity which the force of gravity would impart to W in one second if it could fall vertically. Now, since forces are proportional to the velocities they can impart in a given time,

$$\frac{F}{W} = \frac{v}{g}, \qquad \text{or,} \qquad F = W\frac{v}{g}, \qquad (1)$$

in which g has the constant value 32.16 feet per second. This is the fundamental formula expressing the law of resistance of inertia and giving the horizontal force F that acting constantly for one second will generate the velocity v in a body whose weight is W.

Now suppose a small particle W, at a distance z from the center of the turntable. In passing from the position AA to the position BB this particle moves through the space $\frac{1}{4}\pi z$ in the time $\frac{1}{2}t$. Its mean velocity is hence $\frac{1}{2}\pi z/t$, and its velocity when it reaches the position BB is $\pi z/t$. As this velocity is acquired in $\frac{1}{2}t$ seconds, the velocity v acquired in one second is $2\pi z/t^2$. The force required to produce this velocity is

$$F = W\frac{v}{g} = \frac{2\pi Wz}{gt^2}.$$

Let P be a force, acting at a distance r from the center, which actually puts the bridge into motion; then by moments about the center,

$$Pr = \Sigma Fz = \frac{2\pi}{gt^2} \Sigma Wz^2,$$

in which the summation must be extended over the entire bridge. To do this let x and y be coordinates of W with respect to rectangular axes through the middle of AA and CC; then $z^2 = x^2 + y^2$. Let b be the half width of the bridge, then $w/2b$ is the weight per square foot, and the small weight W may be expressed by $a \cdot w/2b$ where a is an elementary area of the horizontal projection. The last equation now becomes,

$$Pr = \frac{\pi w}{bgt^2}(\Sigma ax^2 + \Sigma ay^2),$$

in which Σax^2 and Σay^2 are the rectangular moments of inertia of the floor surface with respect to the two axes. Inserting the values of these, the equation reduces to,

$$Pr = \frac{2\pi \cdot 2wl(l^2 + b^2)}{3gt^2}, \tag{2}$$

in which $2wl$ is the total weight of the swing bridge, l is the half-length, and b the half-width; l, b, and g should be expressed in feet, $2wl$ in pounds, and t in seconds.

The force P is exerted through the distance $\frac{1}{4}\pi r$ in $\frac{1}{2}t$ seconds, and hence the power required to do this is

$$HP = \frac{\frac{1}{2}\pi Pr}{550t}. \tag{3}$$

When the bridge comes to the position BB it attains its maximum velocity, and is then to slow up and come to rest at CC. The work expended in slowing it up is equal to that expended in putting it into motion, but part of this is furnished by the frictional resistances.

For example, take the heavy swing bridge erected over the Harlem river in 1895, whose length is 390 feet, width 60 feet, and total weight about 2500 net tons. Here $l = 185$ feet, $b = 30$ feet, and $2wl = 5\,000\,000$ pounds. The time of opening was specified as $1\frac{1}{2}$ minutes, or $t = 90$ seconds. From (2) the value of Pr is,

$$Pr = \frac{6.28 \times 5\,000\,000(185^2 + 30^2)}{3 \times 32.2 \times 90^2} = 1\,409\,000,$$

and then from (3) the power required is

$$HP = \frac{1.57 \times 1\,409\,000}{550 \times 90} = 44.7 \text{ horse-powers.}$$

The total power provided for turning the bridge and raising and locking the ends is two 50 horse-power engines, one of which is considered sufficient to operate the structure, although both are generally in use.

Prob. 27. Find, for the above case, the force which is exerted at the circumference of the rack circle in order to overcome the resistance to inertia in opening the bridge, the diameter of the rack circle being about 56 feet.

CHAPTER III.

CANTILEVER BRIDGES.

ART. 29. FUNDAMENTAL PRINCIPLE.

The word "cantilever" originally meant a projecting arm or bracket. A cantilever beam is a beam horizontally fixed at one end and without support at the other. If a beam be laid on two supports, and have the ends projecting beyond the supports, those ends may be called cantilever arms. Thus in the draw bridge of Fig. 40 the arms BM and MC are cantilever trusses, and the structure is to a certain extent a cantilever bridge.

The idea of the cantilever bridge was first developed in the attempt to avoid the disadvantages of continuity (Art. 12). In a continuous bridge a slight elevation or depression of one support causes great changes in reactions and stresses. If, however, the chords be cut near the inflection points for full load, the inflection points for partial load will occur there also, and thus the reactions will be statically determinate. In a continuous truss of three spans, as in the first diagram of Fig. 44, there are four inflection points, and RITTER proposed about 1860 to cut one chord at these points, thus producing the arrangement shown in the second diagram. This proposed truss involves the idea of the modern cantilever structure, but it is defective in being cut at too many places.

A truss of three spans has four supports. For any load there are four reactions, and to determine these four conditions are necessary. Two conditions are furnished by the principles of statics, namely, that the sum of the vertical forces is zero and that the sum of their moment is zero about any center. By hinging the chords in two places, as in the third and fourth diagrams of Fig. 44, two additional conditions are established, since for each of these hinges the moment is zero. These forms are thus statically determinate with regard to reactions and stresses, so that a slight change of level in one support produces no change in them. They have, however, the advantages of continuity in respect to the distribution of stresses,— since the shear and moment diagrams for full load are the same as for the continuous truss. Thus the principal advantage of the continuous system is supposed to be preserved while its greatest disadvantage is entirely avoided.

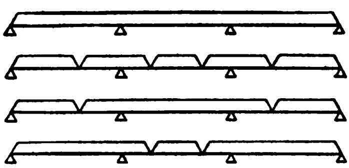

Fig. 44.

The Boyne viaduct, built in Ireland in 1855, was a continuous lattice truss of three spans, the middle span being 267 feet and each side span 141 feet. After erection the upper chord in the middle span was cut at two points 170 feet apart and equidistant from the piers; as some compression was found to exist at these points, the ends of the side spans were lowered slightly. The object of cutting the chords is not clear, since it is stated that they were riveted together again. It was, however, apparently recognized that under a full uniform load the bridge needed no upper chords near the places where they were cut. See Proceedings of Institution of Civil Engineers, Vol. XIV, 1855.

The Kentucky river bridge, built in 1875 by C. SHALER SMITH, has three spans of 375 feet each. It was to be erected over a river gorge 250 feet deep, where false works could not be used, and accordingly the plan of building it out from the shores panel by panel was adopted, the shore ends being securely anchored during the operation. The ends rested upon rock abutments, while the piers were metallic towers about 250 feet high, and hence liable to change in height under temperature stresses. As the structure when finished would be continuous, the disadvantage of the system was apparent, and hence the chords were jointed in each side span at points 300 feet from the shore, thus making a structure like the third diagram in Fig. 44. For illustrations of this bridge see Transactions of American Society of Civil Engineers, November, 1878.

The third diagram in Fig. 44 shows two simple trusses at the ends, and a truss upon two supports with projecting cantilever arms. The fourth diagram shows a simple truss at the middle, and two side trusses with projecting cantilever arms; the modern cantilever bridge is of this type, it being thus called on account of the cantilever arms. It is particularly adapted for use in localities where false works cannot be used in the middle span, the truss being built out from opposite sides panel by panel.

Prob. 28. In the third diagram of Fig. 44 let a be the distance from the end to the hinge, and b the distance between the two hinges. Under a uniform load of w per linear unit, show that each end reaction is $\frac{1}{2}wa$, and that each middle reaction is $\frac{1}{2}w(a+b)$.

Prob. 29. In the fourth diagram of Fig. 44 let a be the length of the first span, b the distance from the second support to the first hinge, and c the distance between the hinges. Show that the reactions for full uniform load are

$$R_1 = \tfrac{1}{2}wa - \tfrac{1}{2}w\frac{b^2+bc}{a}, \qquad R_2 = \tfrac{1}{2}w(a+2b+c) + \tfrac{1}{2}w\frac{b^2+bc}{a}.$$

Draw shear and moment diagrams when $b=\frac{1}{2}a$, and $c=a$.

ART. 30. HISTORICAL AND DESCRIPTIVE NOTES.

The idea of building out cantilever beams from opposite shores of a stream, and then bridging the interval between them by a simple beam, is an old one. An ancient structure of this kind in Japan is described in Von Nostrand's Magazine for January, 1886, and one in Thibet of 112 feet in length is illustrated in THOMAS POPE's Treatise on Bridge Architecture, published at New York in 1811. This curious book of POPE is largely devoted to a design of his own, called the "flying pendant lever bridge," which was to be built out from the opposite shores until the cantilever arms met at the middle, these arms being anchored by huge abutments. With such a structure he proposed to bridge the Hudson river at New York, the span being about 3000 feet. His scheme met with little encouragement, and it was indeed an impracticable one.

The continuous bridges built in Europe from 1850 to 1870, and the discussion of their advantages and disadvantages, led to the development of the cantilever system in the manner described in the last article. Before 1870, however, the subject was approached from another point of view, namely, from that of the suspension system. In a suspension bridge, the truss is supported partly by a cable and partly by inclined stays which are attached to towers. It was proposed by TROWBRIDGE to omit the cable and rely entirely upon the stays; as these could not be conveniently extended to the center of the middle span, he arranged to span the interval there by a simple truss, thus making a construction like Fig. 45. This is the principle of the modern cantilever system, except that the towers and stays are unnecessary. Further information regarding this idea and the influence it exerted is given in Engineering News, December 29, 1883.

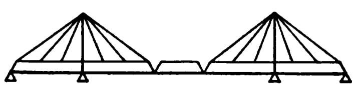

Fig. 45.

A small cantilever bridge was built in 1880 over the Mississippi River at Fort Snelling, Minn., but the history of the practical development of the system really dates from 1883, when the structure over the river near Niagara Falls was completed by C. C. SCHNEIDER. A smaller bridge by the same engineer was also built in 1883 over the Fraser river on the Canadian Pacific Railroad, a sketch of which is given in Fig. 46. Here the end

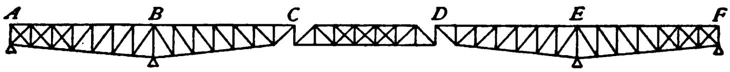

Fig. 46.

spans AB and FE are each 105 feet long, the cantilever arms BC and ED are each 105 feet, and the simple suspended truss CD is also 105 feet. The panels are 15 feet in length, the depth at A and F is 14 feet, the depth at B and E is 25 feet, and the depth of the central truss is 15 feet. The upper chords are really extended across the spaces at C and D, but they are provided with slotted joints so that no stresses can be transmitted by them. At A and F, the trusses are tied down to the abutments by anchor rods.

One-half of the Niagara bridge is shown in Fig. 47. The anchor span AB_1 is 195 feet, the pier panel B_1B_2 is 25 feet,

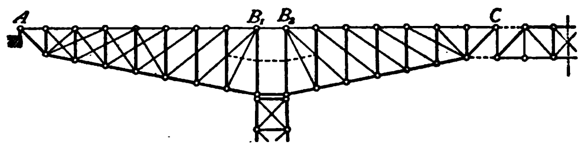

Fig. 47.

the cantilever arm B_2C is 175 feet, and the suspended truss is 120 feet, making a total length of 910 feet. It carries two railroad tracks. The pieces shown in broken lines over the pier are inserted as stiffeners to the posts; those at C are jointed so that they cannot transmit stresses. In erection, false works were used under the end spans, while the cantilever arms and central truss were then built out panel by panel from the piers until they could be joined at the middle. The erection was completed on the Canadian side in 73 days, and on the United States side in 84 days, and the cost of the structure was about one-sixth of that of the suspension bridge of ROEBLING, which then spanned the river near it. Reference is made to SCHNEIDER'S paper in Transactions American Society of Civil Engineers, November, 1885, for a full account of the design and erection.

It would be difficult to record even the names and lengths of the numerous cantilever structures that have been erected since 1883, but a few will be noted in subsequent pages. The longest in the United States is that completed by MORISON in 1892 over the Mississippi river at Memphis, one of whose spans is 790 feet long. The proposed bridge over the Mississippi at New Orleans is to have three spans with a total length of 2274 feet, the middle opening being 1070 feet, and one proposed to cross the St. Lawrence river at Montreal is to have one of the spans 1250 feet in length.

The great Forth bridge in Scotland, the largest bridge in the world, is of the cantilever type, and has two main spans each of 1700 feet, with shore spans of 685 feet. The truss is 350 feet deep over the towers, and of corresponding gigantic proportions throughout. It was completed in 1889 after six years of work. See London Engineering, Dec. 6, 1889, for detailed description and illustrations.

Prob. 30. Consult the engineering journals, find accounts of the following cantilever bridges, and give a diagram of each with the principal dimensions: (*a*) The St. John's bridge, 1885; (*b*) the Louisville and New Albany bridge, 1886; (*c*) the Poughkeepsie bridge, 1887; (*d*) the Sukkur bridge, 1888; (*e*) the Philadelphia bridge, 1889; (*f*) the Red Rock bridge, 1890; (*g*) the Memphis bridge, 1892; (*h*) the New Orleans bridge, 1899.

ART. 31. CLASSIFICATION.

A cantilever structure may be built as a deck bridge or as a through bridge; the webbing may be of any usual type; it may have riveted or pin connections. All the remarks in Parts I and III regarding the economy of different styles of webbing apply also to the trusses used in the cantilever system. The classification now to be explained is that peculiar to the system; it relates to the number of supports at the pier and to the manner of anchoring the spans.

Cantilever bridges usually have three spans, as in Figs. 46 and 47. The simplest style is that shown in Fig. 46, where the truss is supported at the piers upon a single pin. Figure 47, on the other hand, shows two points of support at the pier, and it will be noticed that there are no diagonals in the panel over the pier; this is to avoid the continuity that would otherwise exist, for if shears could be transmitted through this panel AC would be a continuous girder, and a load on B_2C would cause a negative reaction at B_1. By the omission of the diagonals this is avoided, and the reactions at B_1 and B_2 are both positive for all loads on the cantilever arm or central truss (Art. 20).

In either of these forms a load between the piers causes a negative reaction at the support A, and this is the greater the shorter the length of the shore span AB_1. To balance the negative reaction which may be caused by the live load, it is necessary that the truss should be anchored to the abutment at A. The end span is hence often called an anchor span or anchor arm. Thus in Fig. 46, AB is the 'anchor span,' BC is the 'cantilever arm,' and CD is the 'suspended truss,' while BE is called the 'central span' or the 'cantilever span,' since it contains the two cantilever arms. The cantilever span BE is thus held in position by the two anchor spans AB and EF.

When the cantilever system is to be used for more than three spans, one of the arrangements shown in Fig. 48 is used. In the upper diagram BC and DE are cantilever spans, while AB and EF are anchor spans, and CD is called an 'intermediate

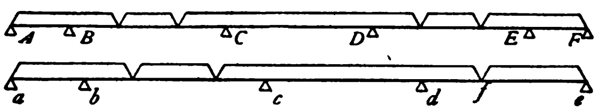

Fig. 48.

span,' because it balances the two cantilever spans on either side; this is the arrangement in the Poughkeepsie bridge, there being, however, three cantilever spans and two intermediate spans. In the lower diagram bc is a cantilever span, ab an anchor span, and cd an intermediate span, while de is partly a cantilever arm df and partly a simple truss fe; this is the arrangement in the Memphis bridge. In Fig. 48 there is shown but one point of support at each pier; two may be used, however, if there be no diagonals in the panel above them.

The suspended truss which connects the ends of the two cantilever arms is merely a simple truss supported at its ends. It evidently receives no stresses except those due to the loads upon its own floor, such being transmitted to the ends of the cantilever arms exactly as if these ends were abutments. Likewise, in the lower diagram in Fig. 48, the simple truss ef is stressed only by the loads that come upon it. It is unnecessary to discuss these simple trusses in the following pages, since Parts I, II, and III treat of them in great detail.

Prob. 31. In the upper diagram of Fig. 48 let a load P be at the middle of BC, and let $AB = l$, $BC = 3\,l$, $CD = 2\,l$; also let the length of the suspended span in BC be $2\,l$. Show that the reactions at the supports A, B, C, D, E, F, are $-\frac{1}{4}P$, $+\frac{3}{4}P$, $+\frac{5}{8}P$, $-\frac{1}{8}P$, 0, 0.

ART. 32. THE CANTILEVER ARM.

In Fig. 49 the simple suspended truss and one cantilever arm are shown; let their lengths be c and b. Under a uniform load of w per linear unit the simple truss transfers $\frac{1}{2}wc$ to each of its ends where it hangs upon the cantilever arm like a concentrated load. At any section in the cantilever arm, distant x from its end, the shear and moment are

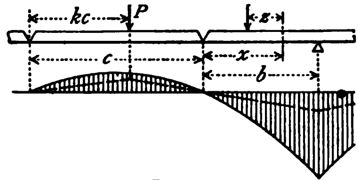

Fig. 49.

$$V = -\tfrac{1}{2}wc - wx, \qquad M = -\tfrac{1}{2}wcx - \tfrac{1}{2}wx^2,$$

and the full line curve in the lower part of Fig. 49 shows the distribution of the moments.

For a load P on the simple truss, at a distance kc from the left end, the amount Pk is transferred to the right end, where it hangs upon the cantilever as a concentrated load. The shear and moment due to this load at any section in the cantilever arm are

$$V = -Pk, \qquad M = -Pkx,$$

and the broken line in Fig. 49 shows the distribution of moments. For a load P on the cantilever arm itself, the shear at any section between it and the pier is $-P$, and the moment is $-Pz$, where z is the distance between load and section.

The shear is always negative for all sections in the cantilever arm, and hence all web members that slope upward toward the pier are in tension. The moment being always negative shows that the upper chord is in tension and the lower in compression. The greatest shear and moment occur at the pier.

At any section in the cantilever arm, the minimum negative shear and moment occur under dead load and the maximum under a full live load. The computation of stresses being made for dead load by the methods of Part I and Part II, those for live load are found by multiplying the former by the ratio of live load to dead. The minimum stresses are those due to dead load; the maximum stresses are those due to both dead and live load.

For example, let Fig. 50 represent a part of a deck cantilever truss, consisting of the suspended truss of four panels and the cantilever arm of four panels. Let the panel length be 15 feet, the depth at M be 16 feet, the depth at G be 19 feet, the dead panel load be 12 000 pounds, or 12 kips, and the live panel load 36 kips. It is required to compute the stresses for LH, HG, Hl, Hh.

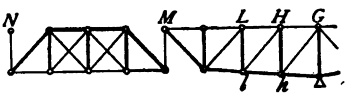

Fig. 50.

Taking dead load first, there are 30 kips at M and 12 kips at each of the other panel points between M and G. The moment for l as a center is $-30 \times 30 - 12 \times 15 = -1080$, and that for h as a center is -1890 kip-feet. As the lever arms of LH and HG are 17 and 18 feet, the dead load stresses for these chords are $+63.53$ and $+105.00$ kips. The difference of these stresses, or $+41.47$ kips, is the horizontal component of the stress in Hl, whence $Hl = 41.47\sqrt{15^2 + 17^2}/15 = +62.68$ kips. The vertical component of the stress in Hl, plus the panel load at H, is the stress for Hh, whence $Hh = -41.47 \times 17/15 - 12.00 = -59.00$ kips.

As the live load is three times the dead load, the live load stresses are three times, and the maximum stresses are four times, the above values. Thus the final maximum and minimum stresses, to the nearest kip, are as follows: $+64$ and $+254$ for LH, $+105$ and $+420$ for HG, $+63$ and $+251$ for Hl, and -59 and -236 for Hh. Here, as always, $+$ denotes tension, and $-$ denotes compression.

Prob. 32. Compute the maximum and minimum stresses for all the members of Fig. 50.

ART. 33. THE ANCHOR SPAN.

In Figs. 51 and 52 are shown one-half of a cantilever bridge of three spans, the former having but one point of support at the pier, while the latter has two, these cases corresponding

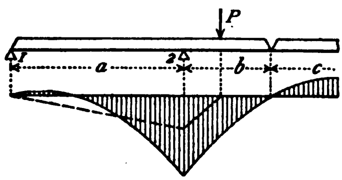

Fig. 51.

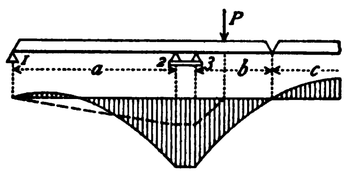

Fig. 52.

to those of Figs. 46 and 47. Below each figure the full line curve shows the distribution of moments for uniform load, while the broken line shows the distribution of moments for a single load P on the cantilever arm. Let the length of the end anchor span *1–2* be denoted by a, that of the cantilever arm by b, and that of the suspended truss by c. The stresses in the anchor span can be computed when the reaction R_1 due to a single load in all positions has been determined.

CASE I, One Support at Pier. — Here R_1, R_2, and P are the vertical forces. The sum of these is zero, and the sum of their moments with respect to the end of the cantilever arm is also zero. Thus the reactions are found to be as follows:

Let P be a load on the anchor span at a distance ka from the left end, where k is a number less than unity. The reactions are

$$R_1 = P(1 - k), \qquad\qquad R_2 = Pk;$$

that is, all loads on the anchor span act exactly as if this were a simple beam or truss.

For a load P on the cantilever arm at a distance kb from the support *2*, the reactions are found to be

$$R_1 = - P\frac{kb}{l}, \qquad R_2 = P + P\frac{kb}{l},$$

and thus all loads on this arm give negative reactions at *1*.

Let W be the total uniform load on the suspended truss; one-half of this is transferred to the end of the cantilever arm, and

$$R_1 = - \tfrac{1}{2}W\frac{b}{l}, \qquad R_2 = \tfrac{1}{2}W + \tfrac{1}{2}W\frac{b}{l},$$

while the other half of W is transferred to the supports on the right.

CASE II, Two Supports at Pier. — Here there are no diagonals in the panel over the pier, and hence no shear can be transmitted through that panel. Accordingly, for a load on the anchor span, $R_1 + R_2 - P = 0$; and for a load beyond the pier, $R_1 + R_2 = 0$. These conditions, in addition to those of statics, determine the reactions.

For a load P on the anchor span at a distance ka from the left end, the reactions are found to be

$$R_1 = + P(1 - k), \qquad R_2 = + Pk, \qquad R_3 = 0,$$

and thus these loads affect the anchor span like a simple truss.

For a load P on the cantilever arm at a distance kb from the support *3*, the reactions have the values

$$R_1 = - P\frac{kb}{l}, \quad R_2 = P\frac{kb}{l}, \quad R_3 = P,$$

and they are independent of the distance between supports *2* and *3*.

For a total uniform load W on the suspended truss $\frac{1}{2}W$ is transferred to its left end, and

$$R_1 = - \tfrac{1}{2}W\frac{b}{l}, \quad R_2 = \tfrac{1}{2}W\frac{b}{l}, \quad R_3 = \tfrac{1}{2}W.$$

These agree with those found from the formulas of the last paragraph by making $k = 1$, and $P = \frac{1}{2} W$.

In comparing the two cases it is seen that the reaction R_1 is the same for both, and hence the stresses in the anchor span are the same in both cases. It is also noticed that R_2, of the first case is always equal to $R_2 + R_3$ of the second case. By considering the values of the reaction R_1 due to the several loads the distribution of live load to give the largest shears and moments for any section in the anchor span is at once derived.

The largest positive shear due to live load occurs when the live load extends from the section to the right hand support *2*. The largest negative shear occurs when the live load extends from the section to the left hand support *1*, and also covers the cantilever arm and the suspended truss.

The largest positive moment due to live load occurs when the anchor arm is fully loaded. The largest negative moment occurs when the cantilever arm and the suspended truss are fully loaded.

These rules imply that the live panel loads are equal. If they are unequal, as in the case of locomotive excess or locomotive wheel loads, the heavier panel loads are to be put as near the section as possible in the case of the shears, and for the positive moments the same position is to be used as for simple trusses (Part I, Art. 61). For the negative moments the locomotive should be put over the end of the cantilever arm and be preceded and followed by the train load.

The greatest negative reaction R_1 is the stress that comes on the anchor rods. If w be the dead load per linear foot, the reaction R_1 due to loads on the anchor span is $+ \frac{1}{2} wa$ and that due to loads on the central span is $- \frac{1}{2} w(b^2 + bc)/a$. If w_1 be the live load per linear foot, then

$$R_1 = \tfrac{1}{2} wa - \tfrac{1}{2}(w + w_1)\frac{b^2 + bc}{a}$$

is the greatest negative reaction. For example, in the Niagara cantilever bridge, $a = 195$ feet, $b = 175$ feet, $c = 120$ feet, $w = 4200$ and $w_1 = 5000$ pounds per linear foot. Then the maximum negative reaction is 878 000 pounds. To resist this, eight anchor rods are provided, each $3\frac{1}{4}$ inches square, giving a cross-section of $84\frac{1}{2}$ square inches; hence the maximum stress upon these rods is nearly 10 500 pounds per square inch.

Prob. 33. When the anchor span is covered with the live load, show that there may be a positive moment when $(w + w_1)a^2$ is greater than $w(b^2 + bc)$, and find the point where the maximum positive moment occurs.

ART. 34. AN ANCHOR TRUSS.

In order to illustrate the method of computing the stresses in the end anchor truss of a cantilever bridge the case of Fig. 53 will be considered, the cantilever arm and suspended truss being the same as that discussed in Art. 32. The anchor span AF is 75 feet, the cantilever arm GM is 60 feet, and the suspended truss MN is 60 feet. The panel length is 15 feet, the depth Bb is 15 feet, and the depth at the pier is 19 feet. The truss is of the Pratt type, where the diagonals can take only tension. The dead panel load is 12 kips, and the live panel load 36 kips, both on the upper chord. It is required to compute the maximum and minimum stresses for the members in the panel $CcdD$.

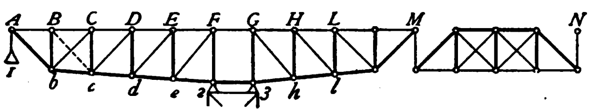

Fig. 53.

The reactions R_1 for each live panel load are first found. The values are as follows, those for the loads from G to N being added together:

Load at	B	C	D	E	G–N	Sum
$R_1 =$	+ 28.8	+ 21.6	+ 14.4	+ 7.2	− 115.2	− 43.2

The live load reaction R_1 is thus -43.2 kips when the entire bridge is covered with live load, and hence the dead load reaction is -14.4 kips. The half panel load at *A* is here left out of account, since its reaction is equal to itself; also the panel loads at *F* and *G* are omitted, as they stress only the posts directly beneath them and produce no reactions at *A*.

For the vertical *Dd* let a section be passed cutting *DE* and *cd*; the minimum stress occurs when *E* is loaded, and the maximum stress when the rest of the bridge is covered. Let p be the panel length; the center of moments for *Dd* is at a distance $15p$ to the left of *A*, and the equations of moments for the two cases of loading are,

$$(14.4 - 7.2)15p + 12(16p + 17p + 18p) + S \times 18p = 0,$$
$$(14.4 + 50.4)15p + 48(16p + 17p + 18p) + S \times 18p = 0,$$

from which -40 and -190 kips are the minimum and maximum stresses for the vertical strut *Dd*.

For the diagonal *Dc* the section cuts *CD* and *cd*; for minimum stress the points *D* and *E* are loaded, and for maximum stress all other points. These stresses are best found by moments, using the method above, thus,

$$(14.4 - 21.6)15p + 12(16p + 17p) - S \times 13.5p = 0,$$
$$(14.4 + 64.8)15p + 48(16p + 17p) - S \times 13.5p = 0,$$

from which $+21$ and $+176$ are the minimum and maximum for *Dc*.

For the chords *CD* and *cd* the lever arms are 17 and 17.95 feet. The dead load moments are -612 and -1188 kip-feet; for live load over the anchor span the moments are $+1620$ and $+1620$; for live load over the central span they are -3456 and -5184 kip-feet. Hence the maximum stresses for *CD* and *cd* are $+239$ and -357 kips, while the minimum stresses are -59 and $+24$ kips. These chords must accordingly be designed to take both tension and compression.

It is thus seen that the computation of stresses in trusses of a cantilever bridge is made by the same principles as for a simple truss, the only difference being that the distributions of live load, and the reactions of the supports, are not the same in the two cases. The graphic methods of Part II may be also applied to draw stress diagrams for dead load and for the different cases of live load, and by combining the values scaled from these the maximum and minimum stresses are obtained.

Prob. 34. Show that the maximum stress upon the anchor rods at A, in Fig. 53, is 129 600 pounds. Compute the maximum and minimum stresses for all the members in the anchor span of Fig. 53.

ART. 35. THE INTERMEDIATE SPAN.

When the cantilever system is applied to four or five spans, the arrangement of Fig. 48 is adopted, and an intermediate span is used to connect two cantilever arms. In this span the chords of the trusses are parallel, and the webbing may be either of the Pratt or Warren type. If the span be sufficiently long, the dead load will overbalance the negative reactions due to live load on the adjacent cantilever arms, and thus anchor

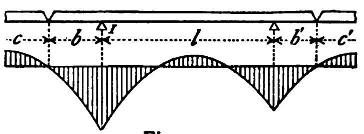

Fig. 54.

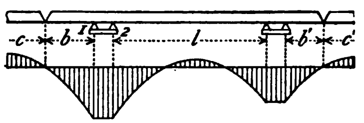

Fig. 55.

rods will not be required to tie the truss down to the piers. In Fig. 54 is shown the case where the truss is supported upon one point at each pier, while Fig. 55 gives the case where each pier presents two points of support, there being no diagonals in the panel above them. The distribution of moments due to dead load is shown in each case by the parabolic curves; if the intermediate span be short, there may be, however, no positive moments in that span under uniform load.

Let l be the length of the intermediate span, b the length of the cantilever arm on the left, and b' the length of that on the right, c the length of the suspended truss on the left, and c' the length of that on the right. The reactions R_1 and R_2, due to a load in various positions, are first to be found, and these are determined by the simplest principles of statics.

Any load in the intermediate span l acts exactly as if that span were a simple truss. A load P at a distance kl from the left end gives $R_1 = P(1 - k)$ for Fig. 54, and $R_1 = 0$ and $R_2 = P(1 - k)$ for Fig. 55. For a uniform load wl, one-half of this is supported at *1* in Fig. 54, and at *2* in Fig. 55.

For a load P on the left cantilever arm at a distance kb from *1*, the reactions at the left pier are found by taking the center of moments at the right pier. Accordingly

$$\text{for Fig. 54,} \quad R_1 = P + P\frac{kb}{l},$$

$$\text{for Fig. 55,} \quad R_1 = P, \quad R_2 = P\frac{kb}{l}.$$

Likewise for a load P on the right cantilever at a distance kb' from its left end, the reactions at the left pier are

$$\text{for Fig. 54,} \quad R_1 = -P\frac{kb'}{l},$$

$$\text{for Fig. 55,} \quad R_1 = 0, \quad R_2 = -P\frac{kb'}{l}.$$

If in these k be made unity and P be made one-half the load on the suspended truss, the reactions due to the loads on the two suspended trusses are found. Thus all loads on the left of the intermediate span give positive reactions at the left pier, while those on the right give negative reactions. By considering these reactions, the rules for placing the live load so as to give the largest shears and moments at any section in the intermediate span are readily determined.

The largest positive shear at any section occurs when the live load covers the truss on the left of *1*, and also the segment between the section and the right pier. The largest negative moment occurs when the live load covers the segment between the left pier and the section, and also the truss on the right of the right-hand pier.

The largest positive moment due to live load occurs when the span is fully loaded. The largest negative moment occurs when the trusses on the right and left of the piers are fully loaded.

The length of the intermediate span required in order that there may be no negative reaction is found as follows: Let the cantilever span on the right be longer than that on the left, or $b' + \frac{1}{2}c'$ be greater than $b + \frac{1}{2}c$. Let w be dead load, and w_1 be live load per linear unit. Then if there be no reaction at *1* in Fig. 54, the dead load on the left of the right pier balances the dead and live load on the right of the pier, or

$$\tfrac{1}{2}(w + w_1)(b'c' + b'^2) - \tfrac{1}{2}wc(b + l) - \tfrac{1}{2}w(b + l)^2 = 0,$$

which gives a quadratic equation for computing $b + l$. For example, take the case where $b = b' = 60$ feet, $c = c' = 100$ feet, $w = 1200$ and $w_1 = 3000$ pounds per linear foot. Then $(b + l)^2 + 100(b + l) = 33\,600$, from which $b + l = 140$, and hence $l = 80$ feet. Thus, in order that R_1 may always be positive, the length l must be longer than 80 feet.

Prob. 35. Deduce for the case of Fig. 55 the equation of condition that the reaction R_1 is zero, and find the length l when $b = 60$, $b' = 65$, $c = 90$, $c' = 100$ feet, $w = 1400$ and $w_1 = 3200$ pounds per linear foot.

ART. 36. AN INTERMEDIATE TRUSS.

Let the intermediate truss in Fig. 56 have a span of 75 feet. On the left is a suspended truss 45 feet long and a cantilever arm 45 feet long. On the right is a suspended truss 60 feet long and a cantilever arm 60 feet long. The intermediate span has five panels each 15 feet long and 19 feet deep. The dead load per linear foot is 12 000 pounds, or 12 kips, and the live

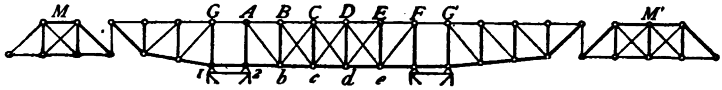

Fig. 56.

load is 36 kips per linear foot, both taken on the upper chord. The diagonals can take only tension, but all other members may take both tension and compression. It is required to compute the maximum and minimum stresses.

The reactions due to the live panel load of 36 kips are first computed, those for loads on MG being added together, as also those for $G'M'$:

Loads at $M-G$		B	C	D	E	$G'-M'$	Sum
$R_1 =$	$+144.0$	0	0	0	0	0	$+144.0$
$R_2 =$	$+64.8$	$+28.8$	$+21.6$	$+14.4$	$+7.2$	-115.2	$+21.6$

Here the panel loads G, A, F, G' are omitted because they produce no stresses except in the posts directly beneath them. The sums give the reactions under full live load, and hence the dead load reactions are $R_1 = +48.0$ and $+7.2$ kips. The truss must accordingly be anchored at *2* so as to take the negative reaction $-115.2 + 7.2 = -108.0$ kips.

For the web members the rule $S = V \sec\theta$ will here apply, since the chords are parallel (Part I, Art. 26). The value of $\sec\theta$ is 1.296 for the diagonals, and unity for the verticals. For each web member the dead load shear is first found, and then the live load shears for the two cases of loading established in the last article.

For the diagonals in the panel $BbcC$ let a section be passed cutting the two chords. The dead load shear is the sum of the dead load reactions at *1* and *2* minus all dead loads on the left of the section, or

$$V = 48.0 + 7.2 - 5 \times 12 = -4.8 \text{ kips};$$

and as this is negative, *Cb* is stressed under dead load. For live load on *MG* and *CF* the reactions are $R_1 = +144.0$, $R_2 = +108.0$, and the shear is

$$V = +144.0 + 108.0 - 4 \times 36 = +108.0 \text{ kips}.$$

For live load on *AB* and *G'M'*, the reactions are $R_1 = 0$, $R_2 = -50.4$, and

$$V = -50.4 - 36.0 = -86.4 \text{ kips}.$$

Thus the largest positive shear is $+103.2$, which gives 133.7 kips for the maximum tension in *Cb*; and the largest negative shear is -91.2, which gives 118.2 kips for the maximum tension in *Bc*. The minimum stress for each diagonal is zero, which occurs when the other one is under tension.

The vertical *Cc* receives its maximum compression when *Bc* has its greatest tension, thus max. $Cc = -103.2$ kips; but its minimum compression is the dead panel load 12 kips, which occurs when *Cb* and *Cd* are not stressed.

For the chord *bc* the first case of loading is live load on *AF*; this gives the shear in the panel as $+36.0$, and the dead load shear is -4.8, as found above. Thus, the resultant shear being positive, *Bc* is under stress, and the center of moments is at *B*. For dead load the equation of moments is

$$M = 48 \times 30 + 7.2 \times 15 - 12(45 + 60 + 75 + 75) = -1512,$$

and for the live load $M = 72 \times 15 = +1080$. Accordingly the combined moment is -432 kip-feet, which gives -22.7 kips as the minimum stress in *bc*.

For *bc* the second case of loading is live load on *MG* and *G'M'*; this gives a resultant negative shear in the panel, which brings Cb under stress, and thus the center of moments is at C. For dead and live loads the moments are

$$M = 48 \times 45 + 7.2 \times 30 - 12(15 + 60 + 75 + 90 + 90) = -1558,$$
$$M = 144 \times 45 - 50.4 \times 30 - 36(60 + 75 + 90 + 90) = -6372.$$

Hence the combined moment for this case is -7930 kip-feet, which gives -417.4 kips as the maximum stress in bc. Accordingly the chord bc is always under compression, whereas in a simple truss the lower chord receives only tensile stresses.

The stresses in the intermediate span are seen to be similar to those in the middle span of a continuous truss of three spans, the upper chords being in tension over the piers. To secure full economy of construction, the length of the intermediate truss for Fig. 56 should be longer than 75 feet in order to increase the positive moments near the middle of the span, and avoid the negative reactions.

Prob. 36. Compute the maximum and minimum stresses for all members of the intermediate truss in Fig. 56.

ART. 37. GRAPHIC METHODS.

As the suspended truss is a simple truss, supported either by two cantilevers or by one cantilever and an abutment, the methods of determining the maximum and minimum stresses in its members are the same as those given in Part II.

The positions of the live load which produce the maximum and minimum stresses in the members of the anchor, cantilever, and intermediate trusses are given in Arts. 32, 33, and 35, and hence no further reference to that subject is needed.

The resolution of the shear, as explained in Art. 50 of Part II, may be conveniently used for the web members after the required bending moments are found. The quotients obtained by dividing the bending moment at each extremity of any diagonal by the length of its horizontal projection may be laid off as ordinates in the same direction if the moments are either both positive or both negative. If, however, one moment is negative and the other positive, as may occur in the anchor and intermediate trusses, the ordinates must be laid off in opposite directions. This requires the diagonal to be produced to meet one of the parallels to a chord member beyond the truss diagram. The resulting form of the force polygon is a double triangle.

Whether the chord stresses are preferably found by this method or by dividing the bending moment by the lever arm of the chord depends upon whether the bracing contains verticals or not.

In order to find the bending moments in the anchor and intermediate spans it is desirable to know the relation which exists between the bending moments of a beam or truss with one or two overhanging ends, and the bending moments of the same truss when the loads on the overhanging ends or cantilevers are removed, and the truss consequently acts as a simple one.

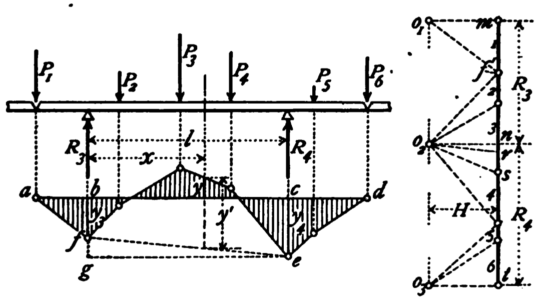

Fig. 57.

In Fig. 57 the loads P_1 and P_6 are transferred to the ends of the cantilevers by the adjacent suspended trusses respectively. After determining the reactions by means of an equilibrium polygon or moment diagram like the upper one in Fig. 16, Art. 11, Part II, the moment diagram in Fig. 57 was constructed by using the three poles o_1, o_2, and o_3, located with the same pole distance directly opposite *m*, *n*, and *t* respectively. This arrangement makes the line *abcd* a single horizontal right line. Let the line *fe* be drawn next. If the loads P_1, P_5, and P_6 on the cantilevers be removed, the truss becomes a simple one whose span is *l*, supporting the loads P_2, P_3, and P_4, and its moment diagram is the one whose closing line or axis is *fe*, all its ordinates being positive. For the section shown the ordinate is y'. If, on the other hand, the truss supports only the loads on the cantilevers, the corresponding moment diagram is *afeda*, all of whose ordinates are negative. On drawing the line *fc*, it is seen that the ordinate at the given section is $-\left(\frac{l-x}{l}y_3 + \frac{x}{l}y_4\right)$. When the truss supports all the loads P_1 to P_6 inclusive, the ordinate at the section is equal to the algebraic sum of the two ordinates whose values are given above; that is,

$$y = y' - \frac{l-x}{l}y_3 - \frac{x}{l}y_4.$$

On multiplying both members of this equation by *H*, and replacing Hy, Hy', $-Hy_3$ and $-Hy_4$ by M, M', M_3, and M_4, respectively, the following required relation is obtained:

$$M = M' + \frac{l-x}{l}M_3 + \frac{x}{l}M_4,$$

in which M' is the bending moment for the simple truss, while M_3 and M_4 are the moments of the given truss at its supports. In substituting numerical values for M_3 and M_4, their signs, which are always negative, must be taken into account. The preceding equation will apply also to an anchor span by making the bending moment at the anchorage equal to zero.

When the chords are parallel, the web stresses are preferably obtained from the vertical shears. Let it be required to find the relation between the shears corresponding to that between the bending moments given above. The rays o_2s and o_2n are parallel to the two sides of the moment diagram cut by the section when all the loads are on the truss, and therefore ns gives the magnitude of the vertical shear V. Now let the ray o_2r be drawn parallel to the closing line fe. The rays o_2s and o_2r are parallel to the two sides of the moment diagram for the simple truss which are cut by the section, and hence rs gives the magnitude of the vertical shear V'. For a similar reason nr gives the magnitude of the vertical shear when the truss supports only the loads on the cantilevers. Let ge be drawn through e parallel to bc; then from the similar triangles o_2nr and egf the value of nr is found to be $-H(y_4-y_3)/l$, it being remembered that the ordinates y_4 and y_3 must be introduced with the negative sign. After substituting M_4 and M_3 for $-Hy_3$ and $-Hy_4$, the required value of the vertical shear is found to be

$$V = V' + \frac{M_4 - M_3}{l}.$$

The preceding investigation, together with the fact that the bending moments are zero at the ends of the cantilever bridge and at the extremities of the suspended truss, lead to the following method of constructing the moment diagram for the entire bridge for any given load.

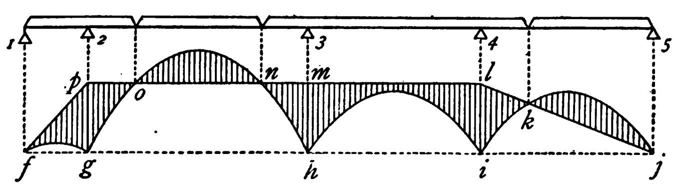

Fig. 58.

Let the bridge shown in Fig. 58 have a uniform load extending over its entire length. On the horizontal axis fj let the parabolic moment-diagrams be constructed under the supposition that the truss for each span is a simple one. Now locating the points *o*, *n*, and *k* directly below the ends of the suspended trusses, these points, together with *f* and *j*, determine the closing lines or axes for the several spans. The line *pm* is first drawn through *o* and *n*, and its extremity joined with *f*, after which the line *jl* is drawn through *k*, and its extremity *l* joined with *m*. On the load line in Fig. 59, *ab*, *bc*, *cd*, and *de* represent the total loads on the respective spans, all the poles are located at the same pole distance, while the rays meeting the load line at *a*, *b*, *c*, *d*, and *e* are parallel respectively to the tangents of the parabolas at the points *f*, *g*, *h*, *i*, and *j*. By drawing through each pole a ray parallel to the closing line of the corresponding span the reactions are obtained as indicated. To find the shear at any point, as at the left end of the suspended truss in the second span, a ray o_2b' is drawn parallel to the tangent to the parabola at *o*, and as o_2b'' is parallel to *pm*, the shear is $b''b'$, and is positive. The moment at any section of the truss is found by multiplying the corresponding ordinate *y* of the moment diagram by the pole distance *H*, the ordinate being measured by the linear scale, and *H* by the scale of force used in laying off the load line. Thus, if *y* be in feet and *H* in kips, the bending moment *Hy* is in kip-feet.

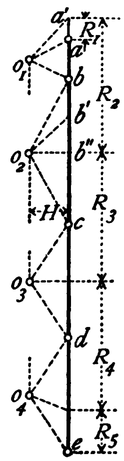

Fig. 59.

If the load consists of equal panel loads, the diagrams will consist of polygons whose vertices lie on the given parabolas in the verticals drawn through the panel points of the loaded chord.

Prob. 37. Find the maximum and minimum stresses in all the members of the anchor truss in Fig. 53.

ART. 38. ECONOMIC LENGTHS.

When a cantilever bridge is to be built, the positions of the abutments and piers will usually be largely determined by local considerations. The most favorable location for a cantilever structure of three spans is a river gorge where the piers are to be built near the water edges, and the abutment anchorages near the top of the banks. If the positions of the two abutments are determined, and the positions of the piers may be more or less varied, the question arises as to what are the proper lengths of the anchor spans, cantilever arms, and suspended truss, in order that the truss may contain the least amount of material.

An approximate solution of this problem may be made by considering only the chords of the trusses and regarding the material in these chords as proportional to the areas of the moment curves. Let Fig. 60 represent one-half of the three-span structure, the length of the anchor arm being a, of the cantilever arm b, and of the suspended truss c. When the anchor span is covered with live load, let A_1 be the moment area due to combined dead and live loads; when the central span is covered with live load, let A, B, C, be the moment areas due to combined dead and live loads. If a negative reaction exists at the left under full uniform load, every ordinate in A will be greater numerically than the corresponding one in A_1. Thus A, B, C, are the moment areas proportional to the material in the chords, and it is required to find the lengths a, b, c, in order that the sum of these areas shall be a minimum. Since the distance L between the abutments is fixed, $2a + 2b + c$ must equal L. Also since the end reaction

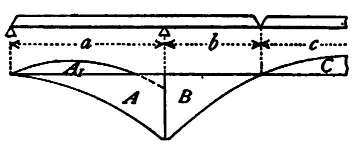

Fig. 60.

under full load is supposed to be negative a^2 must be equal to or be less than $b^2 + bc$, as shown in Art 29.

Let w_1 be the dead and w_2 the live load per linear unit, and w their sum. For the suspended truss the bending moment at any section under full load is $\frac{1}{2}wcx - \frac{1}{2}wx^2$, and the area C, of one-half the moment curve, is,

$$C = \frac{1}{2}w\int_0^{\frac{1}{2}c}(cx - x^2)\,dx = \frac{1}{24}wc^3.$$

Again, for the cantilever arm the bending moment at any section is $-\frac{1}{2}wcx - \frac{1}{2}wx^2$, and the moment area B is

$$B = -\frac{1}{2}w\int_0^b(cx + x^2)\,dx = -\frac{1}{4}wb^2c - \frac{1}{6}wb^3.$$

For the anchor arm the bending moment at any section is $R_1x - \frac{1}{2}wx^2$, where the reaction R_1 is that due to dead load on the anchor span and dead plus live load on the central span. The area A is then

$$A = \int_0^a(R_1x - \frac{1}{2}w_1x^2)dx = \frac{1}{12}w_1a^3 - \frac{1}{4}w(ab^2 + abc).$$

The area C is one of positive moments, while A and B are those of negative moments. In taking the sum all should be regarded as essentially positive, and thus the quantity

$$\frac{1}{24}w(6ab^2 + 6abc + 4b^3 + 6b^2c + c^3) - \frac{1}{12}w_1a^3 \qquad (1)$$

is the expression for the sum of the three moment areas.

Substituting in this expression the value of b in terms of L, namely, $b = \frac{1}{2}(L - c - 2a)$, it reduces to

$$\frac{1}{48}w(L^3 - 3L^2a - 3Lc^2 + 4a^3 + 3ac^2 + 4c^3) - \frac{1}{12}w_1a^3. \qquad (2)$$

Differentiating this with respect to c, putting the derivative equal to zero, and solving, there results $c = \frac{1}{2}(L - a)$, which is the value of c that renders the quantity of material a minimum. On attempting to find the value of a which produces a minimum, it is seen that the function continually decreases as a increases, but that there is no algebraic minimum within the limit imposed. Hence $a^2 = b^2 + bc$ gives the largest allowable value of a in this investigation. Accordingly three conditions are found connecting a, b, and c, and from these

$$a = 0.212\,L, \qquad b = 0.091\,L, \qquad c = 0.394\,L$$

are the lengths which make the above moment sum the least possible. Thus the suspended truss should be about $4\frac{1}{3}$ times, and the anchor span about $2\frac{1}{3}$ times the length of the cantilever arm.

The above results suppose that the end reaction under full load is zero, and it has been shown that if the length a be less than $0.212\,L$, the quantity of material will be greater. It does not follow, however, that a cannot be greater than $0.212\,L$. To investigate this case an expression for the moment sum must be established by regarding the reaction as positive. Without giving the work here, it may be said that a will be found to be about 20 per cent greater than the above value, while both b and c will be smaller.

The problem may also be put in a different way. Let the span l between the piers be given, as also the anchor span a, and let it be required to find the values of b and c that render the chord material a minimum. The expression for the sum of the moment areas is the same as before. In this put $b = \frac{1}{2}(l - c)$, and then differentiate with respect to c, thus deriving

$$b = \tfrac{1}{4}(l - a), \qquad c = \tfrac{1}{2}(l + a),$$

as was first shown by MARBURG. For example, if $a = \frac{1}{3}\,l$, then $b = \frac{1}{6}\,l$, and $c = \frac{2}{3}\,l$. Again, if the anchor arm be fixed at 165 feet, and the central span at 660 feet, the cantilever arm should be 124 feet, and the suspended truss 412 feet; for the Red Rock bridge a is 165 feet and l is 660 feet, while b is 165 and c is 330 feet.

In the above investigation the material required for anchor rods has not been considered; this will tend in practice to make the suspended truss a little longer than the above results. The influence of the webbing will also be appreciable as well as that of the lateral wind bracing. Moreover, the dead load is not perfectly uniform, and the chords cannot vary in section exactly as the moment areas require. For these reasons the above conclusions are to be regarded merely as rough guides in making a design. It is probable, however, that in many cantilever structures the suspended truss has been made too short and the cantilever arms too long for the best economic results. See a valuable paper by MARBURG in Proceedings of Engineers' Club of Philadelphia, July, 1896.

Prob. 38. Prove, if the material in the anchor rods be taken into account, that the economic length for the suspended truss should be longer and that for the cantilever arm be shorter than the values given above. If d be the average depth of truss and h the length of the anchor rods, the value of c, when a and l are given, is $\frac{1}{2}\left(l + a + \frac{hd}{a}\right)$.

ART. 39. DEFLECTIONS.

The deflection at any point of a cantilever truss may be computed by the application of formula (1) of Art. 21, but this involves lengthy numerical work. Treating the truss as a beam of uniform cross-section, approximate expressions for deflection may be derived which will be useful in comparing the stiffness of the cantilever system with that of other structures. The general formulas of Arts. 4 and 13 enable such expressions to be deduced for the end of the cantilever arm. The deflection due to the live load is that required to be found.

In Figs. 61 and 62 let the trusses be regarded as having the constant moment of inertia I and coefficient of elasticity E. Let

1 be the end of the anchor span, *2* the pier, and *3* the end of the cantilever arm. As before, a, b, c, represent the lengths of anchor span, cantilever arm, and suspended truss. Under any given load the point *3* has a deflection Δ, upward in Fig. 61

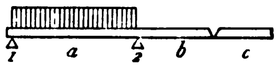

Fig. 61.

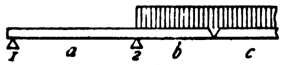

Fig. 62.

and downward in Fig. 62, the shaded areas in these figures denoting the positions of the live load for which the deflection at *3* is desired. As the truss is continuous from *1* to *3*, the theorem of three moments in (8) of Art. 4 applies directly, provided that *3* be regarded as a support which is elevated or depressed the amount Δ above or below the level of *1* and *2*.

CASE I. — Let the anchor span be covered with the live load wa as in Fig. 61. In (10) of Art. 4, each subscript is lowered by unity, and then l_1 and l_2 are replaced by a and b; also h_1 and h_2 are made zero, and h_3 is replaced by Δ. The theorem then becomes

$$M_1 a + 2\,M_2(a+b) + M_3 b = -\tfrac{1}{4}\,wa^3 + \frac{6\,EI\Delta}{b}.$$

Now the load wa produces no moments at *1*, *2*, *3*; hence the first member of the equation vanishes, and

$$\Delta = \frac{wa^3 b}{24\,EI} \qquad (1)$$

is the upward deflection of the end of the cantilever arm.

CASE II. — Let the central span be covered with the live load as in Fig. 62. In (10) of Art. 4, the same changes are to be made as before, and it thus becomes

$$M_1 a + 2\,M_2(a+b) + M_3 b = -\tfrac{1}{4}\,wb^3 + \frac{6\,EI\Delta}{b}.$$

Here, as before, $M_1 = 0$ and $M_3 = 0$, but $M_2 = -\tfrac{1}{2}\,wcb - \tfrac{1}{2}\,wb^2$.

Hence $$\Delta = -\frac{wb^2(4ab + 4ac + 4bc + 3b^2)}{24EI}, \qquad (2)$$

which is the downward deflection of the end of the cantilever arm.

When the Niagara cantilever bridge was completed in 1883, tests of deflection were made; it was found when the anchor span was covered with a load of one gross ton per linear foot per track that the upward rise of the end of the cantilever arm was 1.3 inches, also when both anchor span and central span were covered that the downward deflection of the end of the cantilever arm was 5.4 inches. The value of Δ for Case I was hence 1.3 inches, while for Case II it was $5.4 + 1.3 = 6.7$ inches. To test the formulas the values of Δ will be computed for this bridge, the panel over the pier being not considered. Here $a = 195$ feet, $b = 175$ feet, $c = 120$ feet, $w = 2240$ pounds per linear foot, $E = 25\,000\,000 \times 144$ pounds per square foot. The value of I in the anchor span varies from 130 feet^4 to 1210 feet^4, a mean value being 670 feet^4. Substituting these in the formulas, the deflections are found in feet, and reducing them to inches there are found $\Delta = 0.6$ inches for the first case and $\Delta = -5.8$ inches for the second case, the first being about one-half of the observed value, and the second about one-eighth less.

While the above formulas give only rough values of the deflections for a trussed structure, they are valuable for purposes of general discussion. For instance, it is seen in (2) that the length b is the controlling element, and hence to increase stiffness the cantilever arm should be made short.

Prob. 39. Show by the theorem of three moments that the deflection at the middle of a simple beam, uniformly loaded, is $5\,wl^4/384\,EI$. Also show by the same method that the upward deflection at the middle of the anchor span, due to a uniform load in the central span of the cantilever structure, is $wa^2(b^2 + bc)/32\,EI$.

ART. 40. COMPARISON WITH SIMPLE TRUSSES.

When a distance L is to be spanned by three simple trusses, the most economical arrangement, as far as the superstructure is concerned, is to make the three spans equal in length. If w be the total load per linear foot, the maximum moment at the middle of each span will be $\frac{1}{8}wl^2$, the area of each moment curve will be $\frac{2}{3}l \times \frac{1}{8}wl^2 = \frac{1}{12}wl^3$, and the area of the three moment curves will be $\frac{1}{4}wl^3 = \frac{1}{108}wL^3 = 0.00926\,wL^3$. By deriving similar expressions for the sum of the moment curves for a three-span cantilever structure the relative economy of the two systems may be roughly estimated. A general formula for one-half this sum is given by (1) or (2) of Art. 38, and in using this the term involving w_1 will be omitted as its influence is small, and the neglect partially balances the fact that the alternating stresses in the anchor arm have not been considered.

Let the total length of the cantilever structure be L. Let $a = 0.2\,L$, and let b and c be varied so as to keep the central span at $0.6\,L$. Then for the three following cases s, the sum of the moment areas, has the values as stated:

$$a = 0.2\,L, \quad b = 0.1\,L, \quad c = 0.4\,L, \quad s = 0.0127\,wL^3,$$
$$a = 0.2\,L, \quad b = 0.15\,L, \quad c = 0.3\,L, \quad s = 0.0135\,wL^3,$$
$$a = 0.2\,L, \quad b = 0.2\,L, \quad c = 0.2\,L, \quad s = 0.0153\,wL^3.$$

It is seen that all these values of s are greater than the sum $0.0093\,wL^3$ for the three simple trusses. The smallest value is for the shortest cantilever arm, as already indicated by the previous investigations.

A greater degree of economy can be obtained by making the anchor arm longer; the longest value to which the formulas of Art. 37 apply is $0.212\,L$. For this, the following comparisons are made:

$$a = 0.212\,L, \quad b = 0.091\,L, \quad c = 0.394\,L, \quad s = 0.0114\,wL^3,$$
$$a = 0.212\,L, \quad b = 0.153\,L, \quad c = 0.270\,L, \quad s = 0.0126\,wL^3,$$
$$a = 0.212\,L, \quad b = 0.212\,L, \quad c = 0.152\,L, \quad s = 0.0148\,wL^3.$$

Here the first case shows a moment sum but little exceeding that of the simple trusses, and the same law is again apparent regarding the influence of the cantilever arm.

By making the anchor span still longer, the values of s may be made smaller, but a detailed investigation will show that the lowest possible value of s is $0.0093\,wL^3$, and that this occurs when $a = \frac{1}{3}L$, $b = 0$, $c = \frac{1}{3}L$, that is, when the structure consists of three simple trusses. The cantilever system hence has no theoretic economy over simple trusses when the piers can be located in any position; moreover, when the influence of the alternating stresses in the anchor arm and the material required for anchor rods are taken into account, it is at a marked disadvantage.

In regard to stiffness, the advantage is also on the side of the simple truss. When the distance L is spanned by three simple trusses of equal length, the maximum deflection of each is

$$\Delta = \frac{5\,wl^4}{384\,EI} = \frac{5\,wL^4}{31\,104\,EI} = 0.000161\,wL^4/EI.$$

To compare this with the deflection of the end of the cantilever arm when the central span only is loaded, formula (2) of Art. 39 may be used, and the following results will be found:

$$a = 0.2\,L, \quad b = 0.1\,L, \quad c = 0.4\,L, \quad \Delta = 0.000246\,wL^4/EI,$$
$$a = 0.2\,L, \quad b = 0.15\,L, \quad c = 0.3\,L, \quad \Delta = 0.000569\,wL^4/EI,$$
$$a = 0.2\,L, \quad b = 0.2\,L, \quad c = 0.2\,L, \quad \Delta = 0.001000\,wL^4/EI.$$

These show deflections greater than for the simple truss, the first being 50 per cent higher, the second over four times as great, and the third over six times as great. Moreover, the load on the central span causes an upward deflection of the anchor span; this is less than the downward deflection of the simple truss, the fractions for the three cases above being 0.000063, 0.000084, 0.000100, as computed from the formula given at the end of the last article.

The above comparisons put the cantilever system in the most unfavorable light, because it is supposed that the distance L is divided into three equal spans for the simple trusses. However, if the abutments and piers be fixed by local conditions, as will usually be the case, the cantilever system may have a marked theoretic advantage. Thus, L being given, let it be supposed that the piers are fixed so that a is $0.2\,L$. There will then be three simple trusses whose lengths are $0.2\,L$, $0.6\,L$, $0.2\,L$, and for these the moment sum is

$$s = \frac{w(0.2\,L)^3}{12} + \frac{w(0.6\,L)^3}{12} + \frac{w(0.2\,L)^3}{12} = 0.0194\,wL^3,$$

which is greater than the values found for the corresponding cantilever spans, 70 per cent greater for the first case, 54 per cent for the second, and 31 per cent for the third. The deflection of the middle span of the simple truss system here is $0.000169\,wL^4/EI$, which is very much smaller than those found above for the end of the cantilever arm; here the simple system is preferable as far as stiffness and the absence of injurious oscillations is concerned.

For the second series of comparisons where a is $0.212\,L$, if the piers be fixed by local conditions, the three simple trusses will have lengths of $0.212\,L$, $0.576\,L$, $0.212\,L$, and the moment sum s will be found to be $0.0175\,wL^3$, which is also larger than the values for the cantilever system. In fact, in all cases where the piers and abutments are fixed in position, the cantilever system will be found to have a theoretic advantage in economy of chord material.

The above investigation, like those of the two preceding articles, has omitted so many elements that the numerical results obtained are merely rough guides to assist the judgment of the engineer. The cost of the piers and the erection of the superstructure are important factors in the selection of the system to be used, and the influence of these cannot be reduced to formulas.

Prob. 40. When $a = \frac{1}{4} L$, $b = \frac{1}{4} L$, $c = 0$, compare the deflection of the end of the cantilever arm with that of the middle of a simple truss having a length equal to the central span of the cantilever structure.

ART. 41. GENERAL COMPARISONS.

The cantilever system is not adapted to short spans where the ratio of dead to live load is small on account of the provision that must be made for the alternating stresses in the anchor spans, as well as on account of the high deflections that occur under live load. For a highway structure where the live load is light, it may, perhaps, be sometimes advantageously used for spans of about 300 feet; if built as a through bridge, and the upper chord be made to imitate the curve of a suspension cable, as in the beautiful bridge built by J. M. PORTER at Easton, Pa., it is particularly suitable for city and suburban structures.

The cantilever system is best adapted for use in long spans where the ratio of dead to live load is large, and especially where the simple truss or the arched bridge is difficult to erect. The longest simple trusses are those of 550 and 553 feet over the Ohio river at Cincinnati and Louisville; in these the economic limit of the system is probably nearly reached. For longer spans it will be a question as to whether an arched, a suspension, or a cantilever structure shall be used. For spans between 400 and 550 feet the cantilever system can compete with the simple truss and with the arch. For spans between 500 and 1500 feet it can compete with the arch and with the suspension bridge. For spans exceeding 1500 feet it cannot probably be so economically built as the suspension structure. It is true that the Forth bridge with its great spans of 1700 feet is of the cantilever type, but it is thought by many engineers that the selection of this was a mistake. In any given case local considerations regarding the piers and the difficulties of erection are the elements that mainly control the selection of the system to be used.

In the preceding pages the computation of stresses in cantilever trusses by locomotive wheel loads has not been discussed. These are generally specified and must always be used for the design of the stringers and floor beams, but for the usual cases where the spans are long so that the dead load equals or exceeds the live load, it is unnecessary to use them for the trusses. Two or three panel excess loads at the head of the train may be generally substituted for the locomotive wheel loads to give the stresses with all needed precision. When these are on the cantilever arm it is clear that they should be placed as near the section as possible to give the maximum shear, and at the end of the arm to give the maximum moment. The use of locomotive wheel loads for computing stresses in trusses of long span introduces a hair splitting refinement which is unwarranted by the actual conditions of traffic, and it is noted with satisfaction that the practice seems to be on the decline.

There is no mysterious principle in the cantilever system whereby it is more advantageous than others. As noted in Art. 29 it is merely an adaptation of the continuous system, whereby the defects of the latter are avoided and its theoretic advantages preserved. A continuous structure is theoretically more economic than a series of simple trusses having the same spans; the moments are more uniformly distributed, those over the piers being negative and those near the centers of the spans being positive; the cantilever system preserves this distribution of moments. The continuous structure may often be erected without false works by building it out panel by panel; the cantilever system has the same advantage. The cantilever structure avoids in part the alternating stresses which occur in all spans of the continuous bridge, and it avoids entirely the variations in stresses that may arise by alterations in the levels of the supports.

In conclusion it may be suggested that probably the common three-span cantilever bridge, which has been mainly discussed in this chapter, has a lower degree of economy than the arrangement where the simple trusses are in the end spans, as in the Kentucky river bridge. In this plan no anchorages are necessary and since the intermediate truss is longer than an anchor arm it is influenced less by alternating stresses. For four or six spans this arrangement must necessarily be used at one end, as in the Memphis bridge, and wherever local conditions permit the end spans to be sufficiently long it should receive careful attention as a plan whereby the cantilever structure can best utilize the theoretic economy of the continuous system.

CHAPTER IV.

SUSPENSION BRIDGES.

ART. 42. HISTORICAL NOTES.

Since very early times ropes have been stretched over rivers in order to assist the ferriage of boats, or to carry small parcels across in a suspended basket which was pulled to and fro by a cord attached to it. The next step was for a man to walk across upon the rope, keeping his balance by the help of two other ropes hung somewhat higher so that he could grasp them with his hands. Later two ropes were hung side by side and a rude roadway laid upon them, thus forming a narrow foot bridge. In the eighteenth century chains were used instead of ropes and the structure made sufficiently heavy to allow the passage of animals and vehicles. All suspension structures erected prior to the beginning of the nineteenth century were of this rude type; they were few in number, short in span, and very deficient in rigidity.

The first true suspension bridge was erected by JAMES FINLAY in 1801 at Greensburg, Pa.; it was distinguished from all previous structures by having the roadway nearly horizontal and hung from the chains by vertical rods, while the chains themselves passed over towers and by means of backstays were anchored to the rock. Fig. 63 shows a side elevation of this bridge where *BD* are the stone towers built at the sides of the stream, *BCB* the suspended chains, *BA* the back-

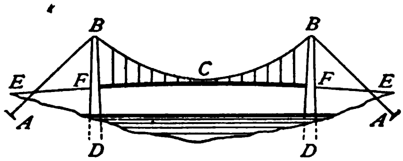

Fig. 63.

stays anchored at A in the rock under the approaches; EF the surface of the approaches, and FF the roadway which is hung from the two parallel cables by the vertical rods. The span of this bridge between the towers was 70 feet, its width 12½ feet, and its cost $6000; it was warranted, all but the flooring, to last fifty years.

Eight suspension bridges of this type were erected by JAMES FINLAY and JOHN TEMPLEMAN prior to 1810; one at Cumberland, Md., had 130 feet span; one near Washington, D.C., had 130 feet span; one at Wilmington, Del., had 145 feet span; and one over the Schuylkill river above Philadelphia had two spans of 148 feet each, with a tower 10 feet wide between them. In 1808 FINLAY was granted a patent for this system of bridge construction, and the knowledge of it was widely spread by the descriptions given by THOMAS POPE in his Treatise on Bridge Architecture, published at New York in 1811.

In these bridges the cables were made of chains, but the fact that iron wire had greater strength was soon recognized. In fact a foot bridge with iron wire cables, having 408 feet span, was erected in 1806 over the Schuylkill river, the platform being probably laid on the cables in the old style. In 1814 TELFORD made investigations and concluded that with wire cables it was possible to build a suspension bridge a thousand feet long. In 1819 BROWN built one with 450 feet span, and in 1826 TELFORD erected one over the Menai straits with spans of 580 feet; these, however, were chain bridges, the chains in the Menai bridge being made of bars 9 feet long, 3¼ inches wide, and 1 inch thick, united at their ends by coupling bolts. Throughout Europe the suspension system gradually spread as an advantageous one for highway structures of long span. In 1834 a bridge of 870 feet span was erected at Freiburg, Switzerland, the four cables of which were made of wire, 1056 wires being used in each cable

In 1842 CHARLES ELLET built a wire suspension bridge across the Schuylkill river which had a span of 343 feet; in 1848 he built one across the Niagara river which was used for highway traffic until the completion of the heavier structure by ROEBLING in 1855. In 1848 he also built one over the Ohio river at Wheeling, which had the great span of 1010 feet; this was blown down in 1854. All suspension structures built between 1810 and 1850 were of the FINLAY type, shown in Fig. 63, the roadway being hung from the cables by vertical rods; to prevent oscillations, however, inclined rods called stays were attached to the roadway at various points and carried to the tops of the towers, while guy rods were run laterally and downward from the roadway and secured to points on the banks of the stream. In spite of these precautions these bridges were subject to violent oscillations in gales of wind and many were destroyed. Even under the passage of ordinary traffic they were liable to great deflections, and it was then generally supposed that the system could not be advantageously adapted to railroad structures.

The Niagara suspension bridge completed in 1855 by JOHN A. ROEBLING marks an epoch in the history of this system, it being the first and only suspension structure which has been built for heavy railroad traffic. The span between towers was 821 feet, the width 15 feet, and it had four cables, each

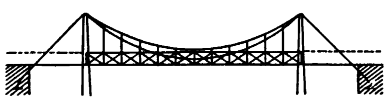

Fig. 64.

10½ inches in diameter and made of 7 twisted strands of wire. The upper deck was for railroad and the lower deck for highway traffic. The distinctive feature introduced was that the roadways were supported by two trusses 16 feet deep, these trusses being hung from the cables by rods. By the use of the truss the stiffness of the structure was greatly increased, this tending to cause a partial load to be uniformly distributed over the cables. This bridge was successfully used for 42 years; in 1880 the wooden trusses were taken out and replaced by steel ones; in 1886 the stone towers were also replaced by steel. In 1897 the bridge was taken down, giving way to the steel arch erected by L. L. BUCK (Art. 87).

The East river bridge between Brooklyn and New York, completed in 1883, has a central span of 1595 feet and two side spans of 930 feet; it has eight cables each about 16 inches in diameter, and carries a foot walk, two wagon roadways and two tracks for light railroad passenger traffic. The second East river bridge, begun in 1897, has a main span of 1600 feet, and will have two foot walks, two carriage ways, four trolley tracks and two elevated railroad tracks. These are the longest suspension structures in the world.

The suspension bridge is adapted to long spans where false works cannot be used. In erecting it the cables are first erected on the towers and anchorages, then the vertical hanger rods attached to the cables, and the truss finally built out from the towers panel by panel and secured to the hangers. The Niagara bridge required three years for its erection, and the Brooklyn bridge thirteen, a large proportion of the time in the latter being spent on the towers and anchorages. With the modern improvements in methods of erection this time can be very much reduced, and it is estimated that the great bridge of 3200 feet span, proposed for the Hudson river at New York, can be built in six or eight years. It is now clearly recognized, for spans exceeding 1500 feet, that the suspension system affords the greatest advantages, and that it may be used for the longest spans and the heaviest traffic with a greater degree of economy than either the cantilever or the arch.

Prob. 41. Consult the article Bridges in the 1896 edition of JOHNSON's Cyclopaedia, and ascertain the size of the cables in the Brooklyn bridge, the number of wires in each and the method by which the cables were made. See also Harper's Magazine, May, 1883; and Van Nostrand's Science Series, No. 32.

ART. 43. STRESSES IN THE CABLE.

An unstiffened suspension bridge is one in which each load is transferred directly to the cables through the hanger rod to which it is hung. It is here proposed to find the stresses in a cable of such a bridge when the entire roadway is uniformly loaded. If there be no loads hung upon the cable it assumes the curve called the elastic catenary; but usually the weight of the roadway and live load is far greater than the weight of the cable and hence it is allowable and customary to regard the entire weight of both as uniformly distributed on a horizontal plane. The letter w will represent this uniform load per linear unit for one cable.

In Fig. 65 let BC represent one-half of a cable of a suspension bridge, C being the middle of the span. Let l be the span and h the sag of the cable below the tops of the towers. The cable BC is held in equilibrium by a horizontal tension H acting at C, a tension T tangent to the curve at B, and the uniform load $\frac{1}{2}wl$ which is distributed over it.

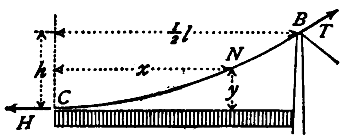

Fig. 65.

Taking moments about B gives

$$Hh - \tfrac{1}{2}wl \cdot \tfrac{1}{4}l = 0 \quad \text{or} \quad H = \frac{wl^2}{8h}, \tag{1}$$

which is the horizontal tension at C. To find T it is noted that this is the resultant of H and the load $\frac{1}{2}wl$, or

$$T = \sqrt{H^2 + (\tfrac{1}{2}wl)^2} = \frac{wl^2}{8h}\sqrt{1 + 16\frac{h^2}{l^2}}, \tag{2}$$

which is the tension in the cable just before it reaches the top of the tower. It is seen that H and T increase as the sag h decreases.

The fraction h/l may be called the sag ratio, as it denotes the ratio of the sag to the span, and it will be designated by s. Using this letter formulas (1) and (2) may be written

$$H = \frac{wl}{8s}, \qquad T = \frac{wl}{8s}\sqrt{1 + 16s^2}, \tag{3}$$

in which wl is the total uniform load on the entire cable. The sag ratio s varies in practice from 1/7 to 1/15. FINLAY in 1810 recommended 1/7, but smaller values have generally been used. In the Niagara bridge the upper cable had about 1/15 and the lower one about 1/11; in the Brooklyn bridge the value of s is about 1/12.5. When s is small the stresses H and T become great, and hence, as far as the cables are concerned s should be a large fraction; this, however, requires high towers and the value of s to be selected in any case should be that which renders the total expenditure a minimum.

The curve of the cable under a uniform load is a parabola. To show this let N in Fig. 65 be any point whose coordinates with respect to C are x and y. Taking moments about N, and substituting the value of H from (1), there results,

$$y = \frac{wx^2}{2H}, \qquad y = \frac{4h}{l^2}x^2, \tag{4}$$

which is the equation of the common parabola. It is also seen geometrically that the curve is a parabola by noting that the forces H, wx, and the inclined tension at N meet at a point distant $\frac{1}{2}x$ from C.

The tension in the cable increases from C to B and its constant horizontal component is H. At any point N, distant x from C, the tension in the cable is

$$T_x = \sqrt{H^2 + (wx)^2} = H\sqrt{1 + 64\frac{h^2x^2}{l^2}}, \tag{5}$$

and this becomes equal to T when $x = \frac{1}{2} l$. When the cables are made of chains or eye bars it is possible to vary the sections so that they are proportional to the stresses that come upon them. This method of constructing cables is no longer employed, since greater economy can be attained by the use of steel wire whose strength is far greater and more reliable than any system of chains or links. With wire, however, the cross-section of the cable must be uniform throughout, and hence it must be proportioned to resist the stress T as given by (2). Steel wire has been made having an ultimate tensile strength of 300 000 pounds per square inch, and a quality suitable for suspension cables can be obtained in the market which has an ultimate strength of 225 000 pounds per square inch. When used in the form of twisted ropes the ultimate strength per square inch of actual metal may be as high as 180 000 pounds per square inch.

When a cable is made of parallel wires let d be the diameter of a wire and N their number, then the diameter of the cable will be approximately $d\sqrt{1.3\,N}$, or the gross area of the finished cable will be approximately 30 per cent greater than the net area of metal. When the cable is made of twisted ropes the gross area exceeds the net area in a much higher proportion.

The backstay BA in Fig. 63 should have the same inclination to the tower as the cable does, otherwise there will be a horizontal component which will tend to overturn the tower; the stress in the backstay is then the same as the maximum stress in the cable. Sometimes the roadway extends beyond the tower over the distance FE and is hung to the backstay which then takes a curve like the cable; this is the case in the Brooklyn bridge which has two side spans of 930 feet each. Usually the backstays and central cable form one continuous cable extending from anchorage to anchorage and resting at the tops of the towers upon movable saddles.

While the above formulas strictly apply only to a suspension bridge which has no truss, they are commonly used for all cases of full uniform load. As an example let it be required to find the size of a cable for a highway bridge of 520 feet span where the dead and live load per linear foot is to be 1600 pounds and where two cables are to be used having a sag of 45 feet. The load per linear foot for one cable is $w = 800$ pounds, and by (1) the value of H is 600 900 pounds; then from (2),

$$T = 600\,900 \times 1.058 = 635\,800 \text{ pounds}$$

which is the maximum stress in the cable. Taking the working stress for the steel wire as 40 000 pounds per square inch the net section needed is 15.9 square inches. If the cable is made of parallel wires the gross section will be about 20.7 square inches which corresponds to a diameter a trifle larger than $5\frac{1}{8}$ inches. The weight of this cable will be 54 pounds per linear foot.

The stresses in a cable due to its own weight may be computed from the above formulas with sufficient precision if the sag ratio s is less than $\frac{1}{8}$. For the above example taking $w = 54$ pounds, H and T are found to be 40 600 and 42 900 pounds, and thus the greatest unit-stress in the cable, due to its own weight, is 2700 pounds per square inch. The unit-stress due to the weight of the cable is evidently independent of its cross-section; thus if the cable be taken one square inch in section, $w = 3.4$ pounds per linear foot, and $T = 2700$ pounds which is here also the stress per square inch.

Prob. 42. Compute the stresses in a cable due to its own weight when the span is 1000 feet and the sag ratio $\frac{1}{16}$.

Prob. 43. A light foot bridge used in the erection of the Brooklyn bridge had a span of 1595 feet, and the maximum uniform load upon it was 124 000 pounds. There were two cables, each having an ultimate strength of 636 000 pounds, the sag being $73\frac{1}{4}$ feet. Compute the tensions H and T, and the factor of safety of the cables.

ART. 44. DEFLECTION OF THE CABLE.

When the cable is being manufactured it is under stress only from its own weight. After it is completed and the uniform load is hung upon it, this load produces stresses and elongations in the cable that cause it to deflect. During manufacture, therefore, it is to be hung higher than its final position; in order to find the first position the deflection of the middle of the cable due to the uniform load on the roadway is to be computed. Let a suspension cable of the span l have the sag h when under its full uniform load, let w_1 be the weight of the cable per horizontal linear unit, w_2 that on the roadway, and $w = w_1 + w_2$ the total load per linear unit per cable. It is required to find the upward rise of the middle of the cable if the load w_2 be entirely removed.

Let c be the length of the cable when it hangs in its final position with the sag h; referring to Fig. 65 this length is

$$c = 2\int_0^{\frac{1}{2}l}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx,$$

and from the equation of the curve given by (1) of the last article,

$$\frac{dy}{dx} = \frac{wx}{H} = \frac{8hx}{l^2};$$

inserting this, expanding the binominal, and integrating, there results

$$c = l\left(1 + \frac{8}{3}\frac{h^2}{l^2} - \frac{32}{5}\frac{h^4}{l^4} + \ldots\right),$$

or,

$$c = l\left(1 + \frac{8}{3}s^2 - \frac{32}{5}s^4 + \ldots\right), \qquad (1)$$

in which s is the sag ratio. By this formula the length of the cable between the tops of the towers can be computed.

Suppose that all load be removed from the cable except its own weight, then the cable length c shortens to c_1 and the sag h

decreases to h_1. Let δc denote the change in length $c - c_1$, let A be the net area of the cable cross-section and E the coefficient of elasticity of the wire. Let T_x be the stress at the point N in Fig. 65 which is constant over the length dc. The elongation of this element is

$$\lambda = \frac{T_x\, dc}{AE},$$

and the sum of all the values of λ is the total change in length δc, or

$$\delta c = 2\int_0^{\frac{1}{2}l} \frac{T_x\, dc}{AE}.$$

Inserting the value of dc as given by the first and second equations of this article and the value of T_x as given by (5) of Art. 43, and integrating, there results

$$\delta c = \frac{Hl}{AE}\left(1 + \frac{16}{3}s^2\right), \tag{2}$$

which is the total elongation of the cable; and if H be taken as due to the total uniform load on the floor the change in length of cable caused by this load can be computed. Then

$$c_1 = c - \delta c, \tag{3}$$

is the length of the cable when stressed only by its own weight.

Now when the cable shortens to c_1 the sag decreases to h_1 and $h_1 = s_1 l$ where s_1 is the new sag ratio. The relation between c_1 and s_1 is the same as that between c and s in (1), and hence

$$s_1^4 - \frac{5}{12}s_1^2 + \frac{5}{32}\left(\frac{c_1 - l}{l}\right) = 0, \qquad h_1 = s_1 l, \tag{4}$$

are equations from which s_1 and h_1 are determined. Finally

$$\delta h = h - h_1, \tag{5}$$

is the deflection of the cable due to the total uniform load on the roadway.

To illustrate the method, take the numerical example of the last article where $l = 520$ feet, and $h = 45$ feet under full load;

also $A = 15.9$ square inches and $E = 30\,000\,000$ pounds per square inch for steel wire. Here $s = 45/520 = 0.086538$, $s^2 = 0.007491$, $s^4 = 0.000056$ and from (1) the cable length c is 530.201 feet. As $H = 600\,900 - 40\,600 = 560\,300$ pounds, (2) gives $\delta c = 0.634$, and accordingly from (3) the length c_1 is 529.567 feet. Next from (4) the value of s_1 is found to be 0.083774, whence $h_1 = 43.56$ feet, which gives the sag that the cable must have during manufacture. Finally, the deflection of the cable from that position under full load is 1.44 feet.

Prob. 44. A suspension bridge has a span of 1000 feet and a sag of 80 feet when fully loaded. The unit-stress H/A due to uniform load on the roadway is 39 000 pounds per square inch. Compute the cable lengths c and c_1, the sag h, and the deflection δh.

ART. 45. APPROXIMATE METHODS.

In the two preceding articles an approximation has been introduced by considering that the weight of the cable is uniform per horizontal linear unit. Further approximations may, however, be safely made, particularly when the sag ratio s is less than $1/12$. One of these is based upon the principle that whenever a is a small fraction $(1 + a)^2$ is practically equal to $1 + 2a$, and $(1 + a)^{\frac{1}{2}}$ is equal to $1 + \frac{1}{2}a$; or if a and b be small fractions $(1 + a)(1 + b)$ is equal to $1 + a + b$. Or again, $1/(1 + a) = 1 - a$.

For instance, in formula (3) of Art. 43 the sag ratio s is small and s^2 is still smaller; thus $\sqrt{1 + 16s^2}$ is nearly $1 + 8s^2$, and the value of T may be expressed as $H(1 + 8s^2)$. If $l = 520$ feet and $h = 45$ feet as in the numerical example of Art. 43 the value of s^2 is 0.007491 and $1 + 8s^2 = 1.060$, while the true value of $\sqrt{1 + 16s^2}$ is 1.058. Thus the approximate method gives values a little too large when used for simplifying roots, and values a little too small when used for simplifying powers and products.

To apply this method to determining an approximate formula for the length of the cable the first and second equations of the last article give

$$c = 2\int_0^{\frac{1}{2}l}\left(1 + \frac{64\,h^2x^2}{l^4}\right)^{\frac{1}{2}}dx = 2\int_0^{\frac{1}{2}l}\left(\frac{1 + 32\,h^2x^2}{l^4}\right)dx,$$

and performing the integrations there is found

$$c = l\left(1 + \frac{8}{3}\frac{h^2}{l^2}\right) = l\left(1 + \frac{8}{3}s^2\right), \qquad (1)$$

which is the same result as would be obtained by dropping s^4 from the expression previously found.

Formula (2) really needs no modification, but it is sometimes more convenient to express δc in terms of T instead of H. Taking $T = H(1 + 8s^2)$ and eliminating H, that formula, by use of the above principles regarding small fractions, reduces to

$$\delta c = \frac{Tl}{AE}(1 - \tfrac{8}{3}s^2), \qquad (2)$$

which gives the cable elongation due to any uniform load that produces the stress T at the steepest inclination of the cable.

Let δh be the change in h due to the change of δc in c; then the sag h_1 when the cable has the length c_1 is

$$h_1 = h - \delta h. \qquad (3)$$

An approximate formula for δh may be found by regarding δc and δh as differential lengths; thus if c in (1) receives the change δc, the sag h receives the change δh, and hence by differentiating (1), and solving,

$$\delta h = \frac{3\,l}{16\,h}\,\delta c = \frac{3}{16}\frac{\delta c}{s}, \qquad (4)$$

from which the deflection δh may be directly computed.

The data of the last article will now be used in these approximate formulas. First the cable length c is found to be 530.387 feet instead of 530.201. As $T = 653\,800 - 43\,700 = 610\,100$ pounds from Art. 43, the above formula (2) gives $\delta c = 0.652$ feet instead of 0.634, so that c_1 is 529.753 feet instead of 529.567. Then by (4) the deflection δh is found to be 1.41 feet instead of 1.44, and finally the sag h_1 is 43.59 feet instead of 43.56. It will be noticed, while the cable lengths computed by the two methods differ somewhat, that the changes in length and the deflections practically agree. Considering the uncertainty in E it may be concluded that the formulas of this article are sufficiently precise for all preliminary numerical computations, and indeed for nearly all practical cases.

Prob. 45. A suspension bridge has a span of 1000 feet and a sag of 100 feet when fully loaded. The total uniform load, exclusive of weight of cable, is 1 200 000 pounds per cable, and the net cross-section of the cable is 40 square inches. What is the sag of the cable before the load of 1 200 000 pounds is applied?

ART. 46. EFFECT OF TEMPERATURE.

When the temperature rises the cable elongates and its sag increases; when the temperature falls the cable contracts and its sag decreases. The formulas and computations of the preceding articles may then be regarded as for a standard temperature which will be taken at 50 degrees Fahrenheit. The inquiry will now be made as to the effect produced in the cable by a rise of t degrees in temperature.

Let c be the length of the cable at the standard temperature, h its sag, and s its sag ratio. Let δc be the elongation of the cable under a rise of t degrees; if ϵ be the coefficient of expansion, or the elongation of a unit-length under a change of one degree, then

$$\delta c = \epsilon t c, \tag{1}$$

and from (4) of the last article the deflection is

$$\delta h = \frac{3}{16}\frac{\epsilon tc}{s}. \qquad (2)$$

If the temperature rises, t is taken positive, and δh being positive denotes a downward deflection; if the temperature falls t is taken negative and δh is then an upward deflection.

When the cable falls the stress in it decreases and when it rises the stress increases. Approximate expressions for the change in stress may be obtained by differentiating (1) and (2) of Art. 43 with respect to h, and letting δh, δH and δT be finite differences. Thus,

$$\delta H = -\frac{wl^2}{8h^2}\delta h, = -H\frac{\delta h}{h}, \qquad (3)$$

which gives the change in horizontal tension due to a rise of t degrees; also

$$\delta T = -(1 - 16s^2)\,T\frac{\delta h}{h}, \qquad (4)$$

which gives the change in the tension at the towers for a rise of t degrees.

It is seen from (3) that the relative change in horizontal stress is the same as the relative change in deflection; thus, if h changes two per cent H also changes two per cent. From (4) it is seen that the change in T is also proportional to the change in deflection. When the temperature rises the minus sign shows that H and T are decreased; when it falls they are increased.

For example, let a cable 1000 feet in span have a sag of 100 feet, or $s = \frac{1}{10}$ at the standard temperature of 50 degrees. Then from (1) of the last article the length of the cable is $c = 1026.67$ feet. Let the coefficient of expansion of the steel wire be 0.0000072. It is required to find the changes in deflection and stress when the temperature rises to 110 degrees. Here $t = 60$ degrees and δc is found to be 0 444 feet, which is the elongation of the cable. Then from (2) the deflection δh is 0.83 feet, and from (3) and (4) it is found that $\delta H = -0.0083\,H$ and $\delta T = -0.0070\,T$. The stresses in the cable are thus decreased about three-quarters of one per cent when the temperature rises 60 degrees above the standard. When the temperature falls to 10 degrees below zero, $t = -60$ and $\delta h = -0.83$ feet, from which $\delta H = 0.0083\,H$ and $\delta T = 0.0070\,T$, or the cable stresses are increased about three-quarters of one per cent when the temperature falls 60 degrees. The total change in sag for the change of temperature from 100° to − 10° is 1.66 feet, and the total range in stress about 1.5 per cent of its mean value.

It is seen from the above equations that the changes in sag and stress due to temperature are greater for a small sag ratio than for a large one. Thus for the above data let $h = 80$ feet or $s = 1/12\frac{1}{2}$; then $c = 1017.07$ feet, $\delta c = 0.439$ feet, $\delta h = 1.03$ feet, $\delta H = 0.0129\,H$, $\delta T = 0.0116\,T$, so that here the total change in sag is 2.06 feet and the total range in stress is about 2.5 per cent of its value at the mean temperature. The effect of temperature upon the stresses in the cable is therefore always quite small.

Prob. 46. For the data of the last problem compute the greatest stress per square inch in the cable. Compute also the stress per square inch when the temperature falls so that the sag is decreased to 99 feet.

ART. 47. EFFECT OF A SINGLE LOAD.

Let a single load P be upon the bridge at a distance $\frac{1}{2}kl$ from the middle, where k is any fraction less than unity. Let the load be very light compared with the cable, so that the latter still remains a parabola with its vertex at the middle. It is required to find the stresses and deflections due to the load P.

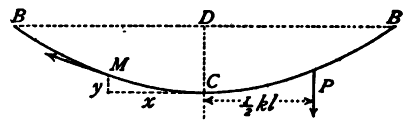

Fig. 66.

As before, let l be the span BB, h the sag DC, and s the sag ratio h/l. The vertical reaction at the left end is $\frac{1}{2}P(1-k)$ and that at the right end is $\frac{1}{2}P(1+k)$, and there is also at each end the horizontal reaction H. To find H take a section at C, and since no moment can exist in a cable, the sum of the moments of the forces on either side of C must vanish; thus

$$H = P(1-k)l/4h \tag{1}$$

is the horizontal tension due to the light load P. Let ϕ be the angle which the curve at any point M makes with the horizontal and T_x the tension at that point; then

$$T_x = H \sec\phi = H(1+\tan^2\phi)^{\frac{1}{2}} = H\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}.$$

But as the curve is a parabola its equation is

$$y = \frac{4hx^2}{l^2}, \qquad \text{and} \qquad \frac{dy}{dx} = \frac{8hx}{l^2}, \tag{2}$$

and hence T_x is known. For the stress T at B make $x = \frac{1}{2}l$; thus

$$T_x = H(1+64h^2x^2/l^4)^{\frac{1}{2}}, \qquad T = H(1+8s^2)^{\frac{1}{2}}, \tag{3}$$

and the tension at every point of the curve, due to P, is determined by inserting in these the value of H from (1).

The elongation of the cable due to P is given by (2) of Art. 44, using for H the value found above, and the deflection at the middle may be written from (4) of Art. 45, or

$$\delta h = \frac{Pl(3+16s^2)(1-k)}{64s^2AE}, \tag{4}$$

in which A is the area of the cable cross-section.

Let h' be the sag of any point M below the line BB and let $\delta h'$ be the deflection of M due to the load P. Since $h' = h - y$, it follows from equation (2) above that

$$\delta h' = \left(1 - 4\frac{x^2}{l^2}\right)\delta h, \tag{5}$$

by which the deflection of any point of the curve may be obtained. This formula (5) holds for uniform load also if the value of δh be substituted for that case.

To find the deflection under the load P, let $x = \frac{1}{2}kl$, and the last equation becomes $\delta h' = (1 - k^2)\delta h$, and then by (4)

$$\delta h' = \frac{Pl(3 + 16s^2)(1 - k)(1 - k^2)}{64s^2AE}. \tag{6}$$

Here if $k = 1$ there is no deflection, and if $k = 0$ the load is at the middle and the maximum deflection obtains. In all these formulas P must be sufficiently light so that the deflected cable still remains a parabola. They are principally useful to show the influence of a single panel load as compared with that of the entire uniform load.

Prob. 47. Show that formula (1) will reduce to the formula for uniform load by substituting $wd(\frac{1}{2}kl)$ for P and then integrating between the proper limits.

ART. 48. HANGERS AND STAYS.

The vertical rods by which the roadway is hung to the cables are called hangers or suspenders. If the roadway is horizontal the lengths of these hangers are readily computed from the equation of the parabola. Usually the roadway is given an upward camber in order to increase the stiffness of the structure, and the camber curve being assumed, the ordinates between this and the parabola give the hanger lengths. The hangers are provided with sleeve nuts at their lower ends or have there other means by which they may be adjusted in length so as to make the tension upon them equal under the uniform load. At the upper end the hanger is usually provided with an eye loop through which a bolt passes to connect it to the lower part of a band that encircles a cable. The hangers are of equal size throughout and proportioned for a tension equal to the maximum floor load that comes upon them.

When stays are used they extend out from the top of each tower to about the quarter points of the roadway, as seen in Fig. 67. These stays relieve the cable of but little stress, their main office being to prevent oscillations in the roadway or truss under wind or unsymmetrical loads. The point of attachment at the foot is near the end of the hanger, and each stay is provided there with the means of regulating its length so that it may have the proper tension. In designing the Brooklyn bridge the proportion of load which the stays were to carry was assumed, the remainder being assigned to the cable, and it was asserted that by the proper adjustment of lengths the stresses in cable and stays would correspond to the calculated ones. Thus for a distance of 133 feet out from the towers the stays were required to carry 1032 pounds per linear foot and the cable 180, for the next 133 feet the stays were to carry 450 and the cable 760, and for the following 133 feet the stays had 188 and the cable 1024, while for 400 feet on each side of the middle of the span there were no stays and the cable carried the entire load of 1212 pounds per linear foot. The computations showed that while the stays supported 22 per cent of the load the tension in the cable was only 12 per cent less than if no stays had been used. The following investigation indicates, however, that it

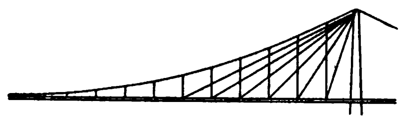

Fig. 67.

is impossible that the stays can carry so great a proportion of the load as assumed above.

If P be the load to be carried by a hanger which is connected with no stay, the stress in the hanger is P. When a stay is connected with the foot of the hanger, let fP be the part of the load carried by the hanger and $(1-f)\,P$ the part carried by the stay, where f is a fraction less than unity.

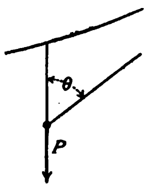

Fig. 68.

Let l_1 and l_2 be the lengths of hanger and stay, and A_1 and A_2 the areas of their cross-sections. Let θ be the angle between the hanger and stay. The deflection of P may be found in two ways, first by regarding it as due to the elongation of the cable and hanger, and secondly taking it as due to the elongation of the stay. As the stress in the hanger is fP, the deflection of the cable is found by writing fP instead of P in (6) of the last article, and then

$$\Delta = \frac{fPl(3 + 16s^2)(1 - k)(1 - k^2)}{64 s^2 AE} + \frac{fPl_1}{A_1E} \qquad (1)$$

is the deflection found by the first method. As the stress in the stay is $fP \sec\theta$, the vertical deflection due to its elongation is

$$\Delta = \frac{(1 - f)Pl_2 \sec^2\theta}{A_2E}. \qquad (2)$$

By equating these two expressions for Δ the value of f may be obtained for a load at any position when A, A_1, A_2, are given.

For example let $l = 1600$ feet, $h = 128$ feet, $s = 0.08$. Let a point be taken 200 feet from the tower or 600 feet from the middle of the bridge, so that $k = 0.75$; also $l_1 = 72$ feet, $l_2 = 236$ feet, $\sec\theta = 1.888$. Substituting these data in the above expressions, equating them and solving for f, there is found

$$\frac{1}{f} = 1 + 1.5753\frac{A_2}{A} + 0.8553\frac{A_2}{A_1}.$$

Now let the cable cross-section A be 134 square inches, suppose that it be required that the stay should carry one-half the load at its foot or $f = 0.5$, also that A_1 and A_2 are to be equal. Then the equation gives $A_1 = A_2 = 79$ square inches which is impracticable and uneconomical. Again let $A = 134$ square inches, and $A_1 = A_2 = 4$ square inches, then the equation gives $f = 0.883$, or the stay carries less than one-eighth of the load. If a stay extends out to the quarter point of the span or $k = 0.5$, a similar investigation will show that it carries less than one per cent of the load at its foot. The true function of stays, then, is not to relieve the cable of stress which they can do but to a very limited extent.

If the point of attachment of hanger and stay be made to deflect by loads at other points, as for instance by the approach of the live load, this causes no stress in the hanger if the bridge be without a stiffening truss, while the stay receives a stress due to its elongation. Let Δ' be the deflection thus caused, then the elongation of the stay is $\Delta' \cos\theta$; if S be the stress in the stay the elongation due to this stress is Sl_2/A_2E. Consequently the stress in the stay is $A_2E\Delta' \cos\theta./l_2$, and the unit-stress is $E\Delta' \cos\theta./l_2$. Accordingly the unit-stresses thus produced are independent of the cross-section of the stay. It thus appears impossible to rationally design a stay to resist the stresses that come upon it.

Stays moreover do not act in unison with the hangers and cables under changes in temperature, while as stiffeners their utility is far inferior to that of a stiffening truss. For all these reasons stays have been practically abandoned in recent designs for suspension bridges, and it is not likely that they will be hereafter used in structures where a truss is used.

Prob. 48. Let $l = 1000$ feet, $h = 80$ feet, $k = 0.5$. Find the approximate lengths of the hanger and stay.

ART. 49. CABLE CONNECTIONS.

Thus far the cables have been supposed to hang in a vertical plane. Usually, however, the cables on each side are drawn inwards so that their distance apart at the middle of the bridge is less than that at the towers, and this is called "cradling the cables." In this operation the cable sag is slightly decreased, and the general effect is to stiffen the structure against lateral oscillations. If b be the amount of this lateral cradling for one cable and h the sag when in a vertical plane, the decrease in sag is

$$\delta h = h - \sqrt{h^2 - b^2} = b^2/2\,h,$$

since b is small compared with h (Art. 44). Thus if $h = 80$ feet and $b = 2$ feet, the decrease in sag is 0.025 feet, and by Art. 43 it is seen that the effect of this is to increase the cable stresses by only three hundredths of one per cent.

At the tops of the towers the cables usually rest on movable saddles in the manner indicated in Fig. 69. The stresses in the cable due to loads in the main span can thus be transmitted to the anchorage as soon as a slight motion of the saddle occurs on its rollers. Before this motion can take place the friction of the rollers on its bed plate must be overcome and thus there will usually be lesser stress in the cable on the land side of the tower when the main span is alone covered with the live load. The difference of the horizontal components of these stresses acts upon the top of the tower and tends to pull it over toward the river side. The coefficient of friction for rollers may be taken roughly at 0.01 and thus the total friction to be overcome before the saddle can move is 0.01 W where W is the total load on the saddle. Thus 0.01 W is the horizontal force acting at the top of the tower which

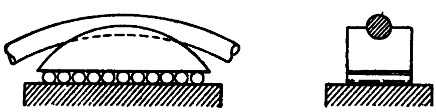

Fig. 69.

must be carefully taken into account in its design; if the tower be a framed structure the stresses due to this may be computed by the method of Part I, Art. 74. For instance, if $l = 520$ feet, $h = 45$ feet, $w = 1600$ pounds per linear foot of cable, as in Art. 43, then $wl = 832\,000$ pounds and $H = 600\,900$ pounds. If there be no land span the weight W is closely 416 000 pounds and the horizontal force required to move the saddle is about 4200 pounds, so that H for the backstay is 596 700 pounds and this is the horizontal tension transmitted to the anchorage.

Another method proposed is to connect the cable directly to the tower on the river side by terminating the ropes in special sockets; on the land side of the tower the backstay is to be connected to the tower in a similar manner. See MORISON's paper in Transactions of American Society of Civil Engineers, December, 1896.

On entering the anchorage the cable wires or ropes are carried around pins at A or are terminated in sockets. These pins or sockets are connected by a series of eye bars with the anchor plate G which is imbedded in the masonry, the eye bars gradually varying their inclination until they become vertical. At the points of change in direction pins connect the eye bars and special blocks of iron and stone are arranged to receive the stress due to the change in angle. After the completion of the metal work within the anchorage the whole is closely surrounded with masonry, concrete, and grout, so that it may be protected from corrosion and the greatest degree of stability be secured.

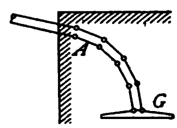

Fig. 70.

The stresses in the anchor bars evidently decrease as they become more inclined to the vertical, so that the upward pull on the anchor plate is much less than the stress in the cable. An exact determination of the upward pull is difficult when the bars are imbedded in the mortar, but the rule given by the following investigation is frequently used, it being supposed that the variation in stress is similar to that which occurs in a belt passing around a portion of a pulley.

Let T_1 be the stress on one side of a belt which is in contact with a pulley over the angle ϕ, let T_2 be the stress on the other side, and T the stress at any intermediate point. In consequence of the friction between belt and pulley T_1 is greater than T_2. Let $d\phi$ be an elementary arc at any point between A and G, then the two stresses in the belt are T and $T-dT$, or the difference of the stresses is dT. If dN be the normal pressure in the direction of the radius and f the coefficient of friction for rest, the law of friction gives $dT=fdN$. But $dN=Td\phi$, and hence

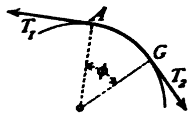

Fig. 71.

$$dT=fTd\phi, \quad \text{or} \quad \log_e T=f\phi+\text{constant}.$$

As T_2 is less than T_1 the angle ϕ must be estimated from the point of contact A. Thus when $\phi=0$, the constant is $\log_e T_1$ and accordingly for any angle of contact,

$$\log_e T_2=-f\phi+\log_e T_1. \qquad (1)$$

Here the logarithms are in the Naperian system and ϕ is in terms of the radius unity. If the logarithms be in the common system and ϕ be in degrees (1) reduces to

$$\log T_2=-0.00758f\phi^\circ+\log T_1, \qquad (2)$$

from which T_2 can be computed when the quantities in the right hand member are given.

It is very uncertain what value for f should be taken for anchor bars imbedded in mortar, but probably 0.8 or 1.0 is not too large. Let a cable enter the anchorage horizontally having a stress of 1 000 000 pounds or 1 000 kips. With $f=0.8$ the following values of T_2 are then computed from (2) for various values of ϕ:

$$\phi = 0°, \quad 30°, \quad 60°, \quad 90°,$$
$$T_2 = 1000, \quad 643, \quad 433, \quad 285 \text{ kips},$$

and thus the stress at the anchor plate where the bars become vertical is only 28.5 per cent of the stress T_1.

Prob. 49. Give all the reductions by which formula (2) is derived from formula (1).

ART. 50. STIFFENING TRUSSES.

In Art. 42 it was mentioned that the lack of rigidity in the early suspension bridges led to the destruction of several of them by wind, and it was shown how the use of a truss to increase the stiffness enabled the system to become practically successful. As its name implies the stiffening truss is not intended to carry loads to the towers but merely to distribute the load uniformly over the cable, and thus prevent the oscillations caused by an unsymmetrical weight or by the wind. A truss is an indispensable adjunct for a modern suspension bridge, and by its help the system can be advantageously applied to the longest spans.

The most common kind of stiffening truss, diagrammatically shown in Fig. 72, is one extending from tower to tower without breaks in the chords. When there are no land spans the truss is needed only between the towers; when land spans exist other trusses are placed over these, and are hung to the cable or supported at their ends only. In these trusses the webbing may be of any convenient kind and the chords are proportioned to take both tension and compression. The hangers are shown connected to the lower chords of the trusses, but if there be two cables on a side hangers may be run from one of them to the upper chord also.

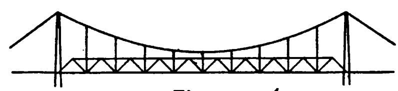

Fig. 72.

Another kind of stiffening truss has its upper chord cut at the middle of the span as seen in Fig. 73, the object in doing this being to reduce the stresses caused by the live load and by changes in temperature, although this is not attained to so great an extent as generally supposed. The Brooklyn bridge has a truss of this kind and at the sliding joint in the middle of the upper chord of the truss a movement of about an inch is observed under the passage of a train while the movement due to the range between winter and summer temperature is about four inches.

Fig. 73.

Other features may also be used in stiffening trusses. They may be made continuous from anchorage to anchorage, or the chords may be cut in the land spans on the plan of the cantilever system. The action of the suspended truss, however, is entirely different from that of a common truss which transfers loads to its points of support. As already mentioned its office is to distribute loads to the cable, and hence under its own dead load it receives no stresses. Even under the action of a live load over the entire span it receives no stresses according to the generally accepted theory. Under the action of a partial live load it is to distribute this load over the cable and in so doing stresses occur in all its members. In the following articles the analysis of these trusses will be given.

Prob. 50. Ascertain the kind of trusses proposed to be used in the great span of 3200 feet across the Hudson river at New York; see Railroad Gazette, Sept. 14, 1894, Engineering News, Sept. 13, 1894, and Transactions American Society of Civil Engineers, Dec., 1896. For the new East river bridge see Railroad Gazette, July 31, 1896; for the proposed third East river bridge see Engineering Record, Jan. 22, 1898.

ART. 51. THE TRUSS WITHOUT HINGES.

The common theory of the suspension truss having unbroken chords over the main span will now be presented. This truss is fastened at the ends to the towers so that either positive or negative reactions may prevail under a partial load. The lengths of the hangers are so adjusted that the dead load is entirely carried to the cable. Under dead load, then, there are no reactions at the end and no stresses in the truss at the standard temperature for which the hangers are adjusted.

When a live load advances upon the bridge the office of the truss is to distribute this load to the cable. If the cable be sufficiently heavy so that it still remains a parabola with its vertex at the middle of the span it follows that the stresses in the hangers are all equal, for the parabola is the curve of equilibrium only under uniform tension in the hangers. Further, if the truss fully distributes the partial live load to the cable, as it is intended it should do, none of it is transferred to the towers, and hence if reactions exist at the ends of the truss the sum of these must be zero.

In Fig. 74 let *1* and *2* be the supports at the towers and let a live load of w per linear unit extend out from the left support the distance z, where z is any fraction less than unity. Let l be the span and w' the uniform upward pull of the hangers per linear foot, so that the total upward pull of the cable on the truss is $w'l$. Let R_1 and R_2 be the reactions of the supports due to the partial load. The fundamental assumptions give

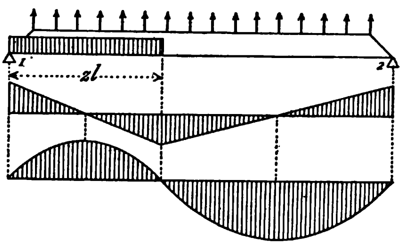

Fig. 74.

$$w'l = wzl, \text{ or } w' = wz, \qquad R_1 + R_2 = 0. \qquad (1)$$

The reactions are found by taking moments about either end, thus

$$-R_2 l - w'l \cdot \tfrac{1}{2} l + wzl \cdot \tfrac{1}{2} zl = 0,$$

and replacing for w' its value wz there results

$$R_1 = \tfrac{1}{2} wlz(1 - z), \qquad R_2 = -\tfrac{1}{2} wlz(1 - z), \qquad (2)$$

and from these the shears and moments at all reactions are found. When $z = 0$, or $z = 1$, the reactions are zero; when $z = \frac{1}{2}$ the reactions have their greatest values $\frac{1}{8} wl$. Hence the maximum shears at the ends obtain when one-half the span is covered with the live load.

The shear at any section in the loaded segment, distant xl from the end, is

$$V = R_1 + w'xl - wxl = \tfrac{1}{2} wl(z - z^2 + 2zx - 2x), \qquad (3)$$

which becomes zero when $x = \frac{1}{2} z$ and $-R_1$ when $x = z$; thus the negative shear at the head of the load equals the reaction. In the same manner it is seen that the shear becomes zero at the middle of the unloaded segment, and that the diagram of shears is as shown in Fig. 74. The maximum shears occur at the ends and at the head of the load, and the maximum maximorum shears obtain when one-half the truss is loaded, their value being $\frac{1}{8} wl$.

The moment at any section in the loaded segment distant xl from the left end has the value

$$M = R_1 xl + w'xl \cdot \tfrac{1}{2} xl - wxl \cdot \tfrac{1}{2} xl = \tfrac{1}{2} wl^2(zx - z^2x + zx^2 - x^2). \quad (4)$$

This becomes 0 when $x = 0$ or when $x = z$, and it has its maximum value when $x = \frac{1}{2} z$, this being the point for which the shear is zero; thus the distribution of moments is that shown in Fig. 74. When $x = \frac{1}{2} z$, max $M = \frac{1}{8} wl^2(z^2 - z^3)$ and as the load advances this attains its maximum maximorum value for $z = \frac{2}{3}$. The greatest positive moment is then $\frac{1}{54} wl^2$ and this occurs when the live load covers two-thirds of the span. In the same manner by writing the equation for the unloaded segment it is found that the maximum moment is at its middle, and that the greatest negative moment is $-\frac{1}{54}wl^2$ this occurring when the live load covers one-third of the span.

In a simple truss the maximum shear is $\frac{1}{2}wl$ and the maximum moment $\frac{1}{8}wl^2$, while for the suspension truss the above investigation gives $\frac{1}{8}wl$ and $\frac{1}{54}wl^2$. Thus the maximum shear for the suspension truss is only one-fourth and the maximum moment only about one-seventh of that for the simple truss. For the simple truss the maximum shears are at the ends; for the suspension truss they occur at the ends and at the middle. For the simple truss the maximum moments are at the middle, for the suspension truss they occur at points distant one-sixth of the span from the middle. Further the simple truss must be proportioned for both dead and live load, while the suspension truss is stressed only by the live load. The suspension truss is hence very light as compared with a simple truss of the same span; the usual practice is to make the webbing uniform in size throughout, designing it for the shear $\frac{1}{8}wl$, and also to make the chords of uniform section throughout, designing them for the moment $\frac{1}{54}wl^2$ and arranging both chords to take either tension or compression.

Such is the common theory of the suspension truss as first presented by RANKINE. For a very heavy cable and a light live load it is not far from correct; for a light cable and a heavy live load the fundamental assumptions do not hold and the distribution of shears and moments is undoubtedly very different from those given by the above analysis. In Art. 53 further remarks regarding the theory of suspension trusses will be given.

Prob. 51. A suspension bridge of 800 feet span is subject to a live load of 2000 pounds per linear foot, and it has two trusses 16 feet deep and with unbroken chords. If the unit-stress for designing the chords be 10 000 pounds per square inch show that the above theory gives 74.1 square inches for the cross-section of the chords.

ART. 52. THE TRUSS WITH CENTER HINGE.

The second kind of stiffening truss where one or both of the chords is broken at the middle is called a truss with center hinge, although there is really no hinge but usually a sliding joint. This truss is fastened at the ends to the towers so that either positive or negative reactions may exist under a partial live load, and the lengths of the hangers are adjusted so that the dead load is entirely carried to the cable. The stresses in the truss are hence caused by live load only.

The fundamental assumptions are the same as before with one exception. The cable is supposed to remain a parabola with its vertex at the middle of the span under a partial load and if this be the case the hangers are equally stressed. It cannot, however, be assumed that the partial live load is entirely distributed to the cable, since the introduction of the center hinge furnishes a condition that there can be no moment at the middle of the truss, and this condition will determine the uniform upward pull of the hangers on the truss.

Let a partial live load extend out from the tower the distance zl where z is any fraction less than $\frac{1}{2}$. Let w be this live load per linear unit and w' the uniform upward pull of the hangers. The reactions being R_1 and R_2 the sum of these is equal to $wzl - w'l$. The sum of the moments of all the vertical forces about either end is zero, and the sum of the moments of the

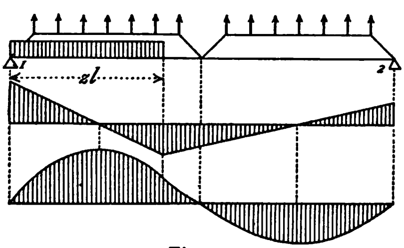

Fig. 75.

forces on one-half the span about the hinge is also zero. These three conditions determine the three quantities R_1, R_2, w', namely,

$$R_1 = \tfrac{1}{2} wl(2z - 3z^2), \quad R_2 = -\tfrac{1}{2} wlz^2, \quad w' = 2wz^2. \qquad (1)$$

The position of the load that gives the greatest value of R_1 is found by putting $dR_1/dz = 0$; this gives $z = \frac{1}{3}$ and max $R_1 = \frac{1}{6} wl$. The greatest value of R_2 occurs when $z = \frac{1}{2}$ and max $R_2 = -\frac{1}{8} wl$. The distribution of shears is as shown in Fig. 75, the shear at the head of the load being $R_1 + w'zl - wzl$ or $\frac{1}{2} wl(4z^3 - 3z^2)$ which has its maximum $\frac{1}{8} wl$ when $z = \frac{1}{2}$.

The maximum moments occur at the points of zero shear. For the loaded segment the shear and moment at a distance xl from the end are

$$V = R_1 - (w - w')xl, \quad M = R_1xl - \tfrac{1}{2}(w - w')x^2l^2.$$

Placing $V = 0$, using the value of xl thus found and inserting also the values of R_1 and w' the expression for the moment becomes

$$M = \frac{1}{8} wl^2 \frac{(2z - 3z^2)^2}{1 - 2z^2}, \qquad (2)$$

which is the maximum value for any given z. The position of the load to give the maximum maximorum moment is found by putting $dM/dz = 0$ which gives the cubic equation $3z^3 - 3z + 1 = 0$; one root of this is $z = 0.395$, and the others are inapplicable on account of being negative or larger than unity. Placing this value of z in (2) the greatest positive moment is found to be $0.1506 \times \frac{1}{8} wl^2$ or about $\frac{1}{53} wl^2$.

For the right hand part of the span the shear and moment may be written for a section distant vl from the right support by taking the forces on the right of the section, thus,

$$V = R_2 + w'vl, \quad M = R_2vl + \tfrac{1}{2} w'v^2l^2.$$

From the first of these it is seen that the shear is zero when

$v = \frac{1}{4} l$, and for this M becomes $-\frac{1}{64} wl^2$ which is the greatest negative moment caused by the live load.

When the live load extends as far as the hinge the reactions become $\frac{1}{8} wl$ and $-\frac{1}{8} wl$ and the tension in the hangers per linear unit is $\frac{1}{2} w$. If the entire span be fully loaded the reactions become zero and the tension in the hangers is w per linear unit. Thus under full live load there are no stresses in the truss.

The theory here given is only correct when the live load is very light so that the cable is not sensibly deformed from a parabolic curve. Strictly speaking the supposition of uniform tension in the hangers under a partial load, or even under a full live load, cannot be realized, and the theory at best is an imperfect one. It is customary to make the webbing of uniform size throughout, designing it for the shear $\frac{1}{6} wl$, and also to make the chords uniform to resist the moment $\frac{1}{53} wl^2$. It is seen that the maximum shear is one-third, and that the maximum moment is between one-sixth and one-seventh, of those for a simple truss of the same span. The truss with center hinge is thus a little heavier than one with unbroken chords.

Prob. 52. Prove that the point of maximum maximorum positive moment is at a distance 0.234 l from the end of the truss.

ART. 53. DISCUSSION OF TRUSS THEORIES.

Notwithstanding the imperfections in the fundamental assumptions of the preceding analysis the theory has been generally used in the design of suspension trusses. The dead load, being usually large compared with the live load, has kept the cable deformations due to the latter from deviating far from the parabolic form, and thus it has been possible to design trusses under the theory which in practice have proved fairly satisfactory.

The imperfection of the theory can be better seen by taking a single load P on the unhinged truss of Fig. 74 at a distance zl from the left end. Then under the given assumption the reactions are $R_1 = P(\frac{1}{2} - z) = -R$. Here if $z = 0$ or $z = 1$, the load comes at one of the supports, one reaction is $\frac{1}{2}P$ and the other $-\frac{1}{2}P$; thus the truss is stressed by a load at one of the supports, which is impossible. In like manner for a single load P on the left hand part of the truss with center hinge in Fig. 75 the reactions are $R_1 = P(1 - 3z)$ and $R_2 = -Pz$; here z cannot be greater than $\frac{1}{2}$; if $z = 0$ the reaction R_1 is P while R_2 is zero. This is as it should be, and it is to be observed that the introduction of the condition imposed by the center hinge has much improved the theory and rendered the truss a more rational one than the structure with unbroken chords.

It is further noticed that in both trusses the reactions are zero under full live load and that the entire load goes upon the hangers. This can scarcely be the case since both cable and truss must deflect under the load and under this deflection it is probable that a part of the load must be carried by the truss, for the deflection of a truss implies that it is stressed. The following investigation of this case indicates that if the truss be very light compared with the cable most of the load will go upon the latter, while if the truss be very heavy compared with the cable but little of the load will be carried by the latter.

Let the entire truss with unbroken chords, as in Fig. 74, be covered with the live load wl. Let fwl be the part of the load that goes upon the cable and $(1 - f)wl$ the part that is carried by the truss, f being a fraction less than unity whose value is to be determined. The deflection of the cable under the uniform load fwl is from Arts. 44 and 45,

$$\delta h = \frac{3}{16}\frac{\delta c}{s} = \frac{Hl(3 + 16s^2)}{16hsAE} = \frac{fwl^2(3 + 16s^2)}{128s^2AE}, \qquad (1)$$

in which A is the cross-section of the cable. Now the deflection of the truss under the uniform load $(1-f)wl$ may be written from the theory of flexure by regarding it as a beam with the constant moment of inertia I. If the chords be of uniform cross-section this is quite allowable, and putting A' for the chord cross-section and d for the depth of the truss the value of I is $\frac{1}{2}A'd^2$. Then,

$$\delta h = \frac{5(1-f)wl^4}{384\,EI} = \frac{5(1-f)wl^4}{192\,A'Ed^2}, \qquad (2)$$

is the deflection of the truss at the middle of the span. As these two values of the deflection must be equal, there is found

$$\frac{1}{f} = 1 + \frac{(9+48s^2)A'd^2}{10s^2Al^2}, \qquad (3)$$

which shows that f depends upon the ratio of the chord section to the cable section and also upon the ratio of depth of truss to length of span. If the ratio A'/A be very small then f is nearly unity or most of the load goes to the cable; if A'/A be very large then f is nearly zero or most of the load is carried by the truss.

For any existing suspension bridge the value of f can be computed from formula (3). For example let $l = 1000$ feet, $h = 100$ feet, $s = 0.1$, $A = 40$ square inches, $A' = 20$ square inches, $d = 30$ feet; then f is found to be 0.70 and $1-f$ is 0.30, or 70 per cent of the uniform live load is carried by the cable and 30 per cent by the truss. The stresses in the truss due to the uniform load $0.30\,wl$ are then computed, the maximum shears being $0.30(\frac{1}{2}wl)$ and the maximum moment 0.30 $(\frac{1}{8}wl^2)$, which are more than double the maximum values given by the theory of Art. 51.

For the truss with center hinge a similar line of investigation may be followed, the deflections of cable and truss being found for the quarter point of the span. If h' be the sag of the cable at this point the equation of the parabola shows that $h' = \frac{3}{4} h$, and thus $\delta h' = \frac{3}{4} \delta h$ in which δh has the value given by (1). For the truss whose span is $\frac{1}{2} l$ the hinge drops the distance δh and the middle the distance $\frac{1}{2} \delta h$ without causing curvature or stress, and thus the actual deflection of the truss at the middle is $\frac{1}{4} \delta h$ where δh has the value given by (1). Again the deflection of a truss with span $\frac{1}{2} l$ is found by placing $\frac{1}{2} l$ for l in (2). Equating the two values there is found

$$\frac{1}{f} = 1 + 4 \frac{(9 + 48 s^2) A' d^2}{10 s^2 A l^2}. \tag{4}$$

Applying this to the same numerical data as before, f is found to be 0.37 and $1 - f$ to be 0.63, showing that the truss carries about five-eighths of the full uniform live load. The maximum shear in the truss is then 0.315 ($\frac{1}{2} wl$) and the maximum moment is 0.158 ($\frac{1}{8} wl^2$), both being about the same as the values given by the theory of Art. 52.

The above discussion indicates that the truss with center hinge is far superior in carrying capacity to the truss with unbroken chords and that its chord stresses are less under full uniform load. As it relieves the cable of load the latter may be made lighter than when a truss without hinge is employed. This discussion, however, is not perfect as it neglects the effect of the load in elongating the hangers and it supposes that all the hangers receive the same tension. When the true theory of suspension structures is developed, it will be found that the hangers are not equally stressed under live load, those near the middle receiving a higher proportion of the load than those at the ends. Under such conditions the curve of the cable is not the common parabola, and the elastic curve in which the truss deflects is not the biquadradic parabola given by the theory of flexure. The expressions for deflection of cable and truss will hence be different from those stated above, but the general conclusion that the value of f depends upon the ratios A'/A and d/l will not be altered. It is not the place in an elementary text-book to dwell upon the examination of doubtful or difficult theories, and perhaps the above discussion has been too lengthy. The student may refer to MELAN's elaborate investigation given in Handbuch der Ingenieurwissenschaften (Leipzig, 1890), as the most complete and correct one yet presented.

Prob. 53. For the above numerical data find an expression for the deflection of the truss with chords at the quarter point of the span, equate it to the deflection of the cable and show that the value of f thus found is less than 0.70.

ART. 54. TEMPERATURE STRESSES IN TRUSSES.

The effect of changes of temperature upon the cable has been discussed in Art. 46. When the cable elongates the truss must deflect downward thus throwing a part of the load upon it; when the cable contracts the truss is deflected upward and is also stressed. An approximate investigation will now be made to determine these stresses in the truss having no hinge.

Let the hangers be adjusted so that at the standard temperature of 50 degrees there will be no stresses in the truss, all the load being carried by the cable. When the temperature rises t degrees the cable elongates the amount ϵtc, where c is the length of the cable and ϵ is the coefficient of expansion. From Art. 44 the deflection of the cable at the middle of the span is

$$\delta h = \frac{3}{16}\frac{\delta c}{s} = \frac{3\,\epsilon tc}{16\,s}, \qquad (1)$$

where s is the sag ratio. This must be equal to the deflection of the truss. Let S be the unit-stress in the chords at the middle of the truss, d the depth of the truss, then from the theory of flexure (Mechanics of Materials, Art. 37),

$$\delta h = \frac{5\,Sl^2}{24\,Ed}, \qquad (2)$$

which is the deflection at the middle of the truss. Equating the values of δh given by (1) and (2) there results, after replacing c by its value in terms of l and s,

$$S = \frac{(9 + 24 s^2) \epsilon t E d}{10 s l}, \qquad (3)$$

which is the unit-stress in the chords at the middle of the truss, due to a rise or fall of t degrees. When the temperature rises this unit-stress is tension in the lower and compression in the upper chord, when it falls the reverse is the case. It is seen that these stresses are independent of the cross-section of the chords and that they increase with the depth of the truss.

For example, let the span be 1000 feet, the sag 80 feet, the depth of the truss 20 feet, the coefficient of expansion 0.0000070, the coefficient of elasticity 30 000 000 pounds per square inch, and let it be required to find the temperature stresses in the chords for a rise or fall of 60 degrees. Here $s = 0.08$, $\epsilon = 0.0000070$, $t = 60$, $E = 30\,000\,000$, $d = 20$, $l = 1000$, and formula (3) gives $S = 2900$ pounds per square inch. As the working unit-stress for the chords of a truss is about 10 000 pounds per square inch, it is seen that here 29 per cent of this is required for temperature stresses, thus leaving only 7100 pounds per square inch to resist the stresses due to live load.

The temperature stresses in the truss are much greater than those in the cable. In the numerical example of Art. 46 it was shown that the variation in cable stress due to a change of 60 degrees in temperature was less than one per cent, and the same holds for the unit-stress. The truss with center hinge is generally supposed to eliminate temperature stresses in its chords, and of course there can be none at the middle. It was however shown by LINDENTHAL in 1888 that such stresses exist and that at the quarter points of the span they are of the same intensity as those at the middle of the truss without a hinge.

The correctness of this view is readily verified by reasoning similar to that in the last article. The sag h' of the cable at the middle of the half-span is $\frac{3}{4}h$, and its deflection $\frac{3}{4}\delta h$; of this $\frac{1}{2}\delta h$ is caused by the drop of the center hinge, leaving $\frac{1}{4}\delta h$ as the true deflection for the truss. Thus $\delta h'$ for the truss is one-fourth of (1), and as the length of the truss is $\frac{1}{2}l$ the value of $\delta h'$ for the truss is also one-fourth of (2); equating these the value of the unit-stress S is the same as given by (3). Thus the truss with center hinge can claim little advantage on account of lower temperature stresses.

The value of S above deduced is for the middle of the span. To determine the other stresses it is only necessary to find the uniform load per linear unit that would produce S and from this compute the other stresses as for a simple beam. Thus,

$$w' = \frac{8A'Sd}{l^2}, \tag{4}$$

where A' is the chord cross-section and d the depth of the truss. In the above numerical example, for the truss without center hinge, let $A' = 20$ square inches and $d = 30$ feet, then $w' = 14$ pounds per linear foot, and thus a temperature change of 60 degrees here produces about the same stresses as a heavy fall of snow. The temperature stresses, it is seen, cannot be decreased by increasing the areas of the cross-sections, but they can be decreased by diminishing the depth of the truss.

Prob. 54. Prove that the maximum shear due to temperature is twice as great in a truss with center hinge as in a truss without one, the depths and chord sections being the same in both trusses.

ART. 55. LIMITING AND PRACTICABLE SPANS.

As the span is increased the cable becomes more highly stressed and a great cross-section is necessary. The longest possible span for a cable carrying no load except its own weight is that for which rupture would occur. Taking the ultimate tensile strength of steel wire at 200 000 pounds per square inch, a cable one square inch in section and weighing 3.4 pounds per linear foot will rupture when the span has such a length L' that the stress in it becomes 200 000 pounds. The expression for this stress is given by (3) of Art. 43, and making $w = 3.4$ and $T = 200\,000$ the limiting span is given by

$$L' = \frac{8sT}{w\sqrt{1+16s^2}} = \frac{470\,590\,s}{\sqrt{1+16s^2}}, \qquad (1)$$

from which the following values of L' are found for different values of the sag ratio,

$$\begin{array}{lllll} s = & \frac{1}{8}, & \frac{1}{10}, & \frac{1}{12}, & \frac{1}{14}, \\ L' = & 52\,610, & 43\,690, & 37\,210, & 32\,320 \text{ feet.} \end{array}$$

Thus for a sag ratio of $\frac{1}{8}$ the limiting span for a steel wire is about ten miles, and for a sag ratio of $\frac{1}{14}$ about six miles.

The limiting practicable span for a steel cable carrying only its weight is that for which the unit-stress reaches the highest allowable limit, say 60 000 pounds per square inch. Using 60 000 instead of 200 000 in (1) these spans are,

$$\begin{array}{lllll} s = & \frac{1}{8}, & \frac{1}{10}, & \frac{1}{12}, & \frac{1}{14}, \\ L = & 15\,780, & 13\,110, & 11\,160, & 9\,700 \text{ feet} \end{array}$$

and thus the limiting practicable span for an unloaded cable is from two to three miles long, depending upon the sag ratio employed.

When the span is shorter than these values of L dead loads may be hung upon the cables, and if it be sufficiently short live loads may pass over the suspended structure. Let w_1 be the weight of the cables per linear foot of span and A their cross-section in square inches. Let w_2 be the total suspended load per linear foot, exclusive of cables. Then using 60 000 pounds per square inch for the maximum unit stress in the cables, T is 60 000 A, and 3.4 A is w_1; hence

$$l = \frac{8s \times 60\,000 A}{(w_1 + w_2)\sqrt{1 + 16s^2}} = \frac{141\,180 w_1 s}{(w_1 + w_2)\sqrt{1 + 16s^2}} \qquad (2)$$

gives the maximum practicable span of a suspension bridge for the loads w_1 and w_2 per linear foot. For example, if the cables have a net section of 536 square inches their weight per linear foot is 1822 pounds; let w_2 be 10 000 pounds per linear foot, then the maximum practicable span for a sag ratio 0.1 is 2020 feet.

Formulas (1) and (2) give no information except that expressed by (2) of Art. 43. The problem of maximum practicable span for a given live load introduces elements of such a complicated nature that general formulation is impossible. The live load being assumed, there is first to be arranged the floor system to support it and the truss to carry the floor, this truss to be stiffened against wind and have due provision made for temperature stresses. The dead load of the truss will be arranged to be carried to the cables by the hangers, but differences of opinion will generally prevail as to the proportion of live load which the cable will take (Art. 53). If the dead load can be roughly expressed as a function of the live load and the span, then w_2 in (2) is a function of l and for an assumed w_1 the maximum practicable span can be computed. This method was followed by the Board of U. S. Engineer Officers of 1894, and its report contains the best solution of the problem thus far made. The live load was assumed at 3000 pounds per linear foot of track, there being six tracks, and the maximum length of train taken as 1500 feet. The analysis gave for the suspended load in pounds per linear foot

$$w_2 = \frac{27\,764\,726}{l} + 13\,605 + 3.24906\,l + 0.00055335\,l^2 + 0.000000003\,l^3. \qquad (3)$$

The cables were taken as sixteen in number, each being $21\frac{1}{4}$ inches in diameter and hence $w_1 = 17\,917$. Inserting w_1 and w_2 in (2) and, using a sag ratio of $\frac{1}{8}$, the solution gives $l = 4335$ feet which is the maximum practicable span for the assumed conditions. The dead load of this structure, exclusive of cables, was found to be 38 386 pounds per linear foot, and the allowable live load, if uniformly distributed, to be 6353 pounds per linear foot. In no system, except the suspension one, is a span of 4000 feet practicable.

Prob. 55. For a railroad suspension bridge having six tracks compute from (3) the weight w_2 for a span of 3200 feet. Then from (2) compute the weight of the cables per linear foot, and if they be eight in number find the net section of each.

ART. 56. UNSYMMETRICAL SPANS.

It sometimes happens, particularly in the design of light foot bridges, that it is desirable to have one tower higher than the other, the roadway being on a heavy grade. In such a case the span is unsymmetrical, as the vertex C of the parabola is not at the middle of the span. The position of the point C is readily determined by the condition that the horizontal component of the cable tension must be the same in all parts of the span.

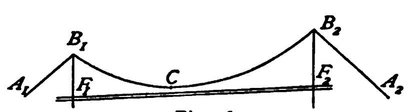

Fig. 76.

Let h_1 and h_2 be the heights of the tops of the towers above the vertex C, and let z be the horizontal distance from the left-hand tower to the vertex. The sag ratios for the two parts of the span then are

$$s_1 = \frac{h_1}{2z}, \qquad s_2 = \frac{h_2}{2(l-z)}, \qquad (1)$$

and the horizontal tensions are by (1) of Art. 43,

$$H = \frac{w(2z)}{8s_1}, \qquad H = \frac{w(2l - 2z)}{8s_2}, \qquad (2)$$

and hence the value of z is

$$z = \frac{l}{1 + \frac{s_2}{s_1}}. \qquad (3)$$

This inserted in (2) gives the horizontal tension H, and then

$$T_1 = H\sqrt{1 + 16s_1^2}, \qquad T_2 = H\sqrt{1 + 16s_2^2},$$

are the tensions in the cable at B_1 and B_2. It is seen that it is impossible to have the same sag ratio for the two parts of the span unless the towers are of equal height.

Suppose the span to be 1000 feet and let it be required that the sag ratio shall be $\frac{1}{12}$ on one side and $\frac{1}{8}$ on the other. Then (3) gives $z = \frac{2}{5}l = 400$ feet, and $l - z = 600$ feet; accordingly from (1) the sag h_1 is $66\frac{2}{3}$ feet and the sag h_2 is 125 feet.

Prob. 56. A foot suspension bridge 800 feet long is designed for a load of 250 pounds per linear foot. The difference in level of the tops of the towers is to be 21 feet, and the sag ratio on the side of the higher tower is to be $\frac{1}{8}$. Find the position of the vertex of the parabola, the horizontal tension H, and the tensions T_1 and T_2.

ART. 57. STIFFENED CABLES.

Many methods have been suggested for stiffening the cables of a suspension structure so as to prevent the oscillations due to wind, live load, and changes of temperature. A system of trussing connecting the cable with the roadway is a natural method of procedure and has been used to a slight extent for short spans. It is, however, generally found more convenient in such cases to make the cable of links or eye bars, and thus the structure is no longer a true suspension bridge. Fig. 77

shows one method of trussing the main span, hinges being used at *B*, *C*, *B*. Here the structure is an inverted three-hinged arch and the computation of its stresses is to be made by the methods of Chap. V.; if the hinge at *C* be omitted it is an inverted two-hinged arch and the analysis in Arts. 87–90 applies to it in all

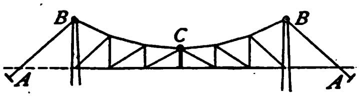

Fig. 77.

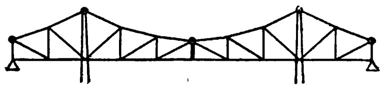

Fig. 78.

respects. When this method of trussing is used for the side spans also, as in Fig. 78, the structure becomes a cantilever bridge with the suspended truss omitted and the methods of Chap. III apply directly to its discussion. It is thus seen that the effect of trussing the cables in this manner is to turn the suspension structure either into the arch or into the cantilever system, the cable forming the upper chord, while the roadway is supported on the lower chord. The upper chord need no longer be a parabola, but may be built to any desired curve, the lower chord receives heavy stresses, and indeed all the distinctive features of the suspension system have disappeared.

Another method of bracing the cable is shown in Fig. 79; here the cable is trussed on its upper side by bracing connect-

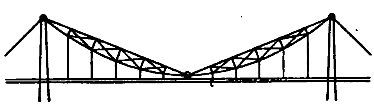

Fig. 79.

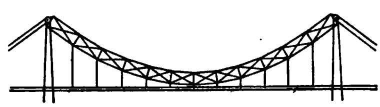

Fig. 80.

ing it with two straight chord members running from the tops of the towers to the middle of the cable. The Point street bridge at Pittsburg, Pa., built in 1877 by EDWARD HEMBERLE, is of this type, the main span being 800 feet in length between centers of towers and the sag 88 feet. The cable is made of eye bars, while the straight upper chords are designed to take both tension and compression; the roadway is attached to the cables by vertical hangers and it is also stiffened by two trusses 8 feet in depth. Under a load of 475 tons distributed over the roadway the deflection at the middle of the span was 4 inches. See Engineering News, July 8, 1876, and April 14, 1877, for detailed descriptions of this interesting structure.

Fig. 80 shows a plan in which there are two parallel cables or chains connected by a system of bracing. Designs for a bridge over the Hudson river at New York have been made on this plan by G. LINDENTHAL, the design of 1894 being for a main span of 3100 feet; the parallel cables are to be 55 feet apart in the vertical plane and made of wire links looped around steel shoes which are joined by pins at the panel points. The sag of each cable is to be 310 feet and the braced cable rib is to be supported on large pins at the tops of the towers. The hangers are 50 feet apart, supporting the floor beams of two decks, which are provided with vertical stiffening trusses and horizontal wind trusses. See Report of the Board of Engineers upon the New York and New Jersey Bridge for detailed descriptions; this report was published in the engineering periodicals of September, 1894, while the report of the Army board appeared in November, 1894.

It is apparent that these methods render the braced cables a structure which involves all the principles of the arch, for an inverted arch differs from the common arch merely in the tensile and compressive stresses being interchanged. Chapter VI will give the analysis by which the stresses may be determined, and it will be seen that the stresses due to changes in temperature are important ones that must receive careful attention.

Prob. 57. See Transactions American Society of Civil Engineers, 1896, Vol. 36, p. 418, for a comparison of different designs for the proposed suspension bridge over the Hudson river at New York. Compare the estimated weights and costs.

ART. 58. CONCLUDING REMARKS.

For spans less than 500 feet the suspension system cannot compare with the simple truss in economy, except for light foot bridges. For ordinary highway traffic short spans may sometimes be used in city parks on account of the æsthetic effect, but these will be more costly than separate trusses and far less rigid. For spans between 500 and 1000 feet the suspension system cannot compete with the cantilever system or with the arch in respect to either economy of construction or stiffness. Between 1000 and 1500 feet, also, the cantilever bridge will usually have the greater degree of economy. Beyond 1500 feet is the field where the suspension bridge has its economic advantages, and as spans of this length will rarely be built, the suspension system will always be one of limited application.

But two suspension bridges for railroad traffic have been completed and used. The Niagara bridge, built in 1854, carried heavy railroad traffic for 43 years, its only fault being its large deflection and the slow speed consequently required in crossing it. The Brooklyn bridge, completed in 1883, was designed to carry only the light passenger traffic of a cable railway and this it has done most satisfactorily. The new East river bridge, to be completed in 1900, will have heavier traffic (Art. 42). For spans of 2000 feet or more, where the ratio of dead to live load becomes large, there can be no doubt but that the suspension system is entirely feasible for heavy railroad traffic, and when the long span of 3200 feet over the Hudson river at New York is built it will be a suspension structure.

Whether the braced cable system is more advantageous for a long span structure than that of unstiffened cables is an open question concerning which different opinions are held. By trussing the cables the stiffness under live load is much increased and the weight of the roadway trusses is decreased;

new elements are however introduced, as the braced cables become inverted arches which are subject to complex stresses arising from temperature and live load, and a comparison of the two plans becomes of the greatest difficulty. The braced system introduces advantages, but it appears also to introduce elements of uncertainty and complexity. The history of the economic development of bridge structures shows that the lines of progress have been in the direction of eliminating uncertain elements and holding fast to those features which secure certainty in the determination of stresses. Thus, double systems of webbing have mostly gone out of use, the cantilever structure has eliminated the uncertainties of the continuous girder, the partially-continuous swing truss has avoided some of the doubts of the older complex draw bridges, and the suspension truss with center hinge is more rational and more effective than one with unbroken chords. If these laws of development continue to hold good for very long spans, the system of bracing the cables and thus introducing complexity does not seem to be in the right direction. But in these long spans a limit is reached to which the general laws of bridge evolution thus far observed may not directly apply, and hence it is not wise to hazard a positive opinion as to the conclusions which may result from the experience of the future.

CHAPTER V.

THREE-HINGED ARCHES.

ART. 59. METALLIC ARCHED ROOFS.

A simple truss under vertical loads has vertical reactions provided one end rests on rollers or on a rocker, so as to allow horizontal motion due to the deflection of the truss and to changes in temperature. When the reactions are inclined under vertical loads, whether this is due to the condition of its supports as in Fig. 81, or to a modification of its form as in Fig. 82, the truss becomes an arch. The loads on the arch produce

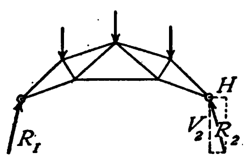

Fig. 81.

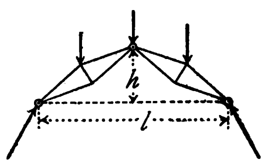

Fig. 82.

a horizontal thrust at the ends equal to the horizontal component of each inclined reaction. The thrust may be resisted either directly by the abutments or by a tie uniting the two supports.

The classification of arches is based on the number of hinges which each truss contains. There are three principal classes containing respectively three, two and no hinges. The simplest form is typified in Fig. 82 and has three hinges, one at each support and one at the crown. When two hinges are employed they are placed at the skewbacks or supports. When the arch has no hinges its ends are usually fixed rigidly to the abutments or piers. When but one hinge is used it is placed at the crown, but as so few arches have been built with a single hinge and as probably no more will be built on account of the theoretic disadvantage of such an arrangement, this class will not be considered. In this chapter arches with three hinges will be discussed.

Three-hinged arches are generally used for railroad train sheds and for exposition buildings of spans that exceed those for which simple trusses may be economically employed. The longest span simple roof trusses in the United States are those of the train shed of the Central Railroad of New Jersey at Jersey City, the span being 142′ 4″ center to center of end pins. Their form is similar to that shown in Fig. 124 of Part II.

The following table gives three of the largest arches yet constructed for train sheds and two of those for exposition buildings:

	Span l.	Rise h.
1. Penna. R. R. at Jersey City, 1892	252′ 8″	90′ 0″
2. P. & R. R. R. at Philadelphia, 1893	259′ 0″	88′ $3\frac{5}{16}$″
3. Penna. R. R. at Philadelphia, 1894	300′ 8″	108′ $5\frac{7}{8}$″
4. Machinery Hall, Paris Exposition, 1889	362′ 9″	149′ 0″
5. Manufactures and Liberal Arts Building, Columbian Exposition, Chicago, 1893	368′ 0″	206′ 4″

A skeleton diagram of the arches of the Broad Street station of the Pennsylvania Railroad in Philadelphia is given in Fig. 83, and an illustrated description showing details of the design may be found in the Railroad Gazette, June 9, 1893, and in Engineering News, June 1, 1893. Those of the station at Jersey City are of the same form and are described in the Railroad Gazette, Oct. 2, 1891, and Engineering News, Sept. 26, 1891. The details of the arches of the Philadelphia and Reading Terminal station are given in Engineering News, Jan. 19 and Feb. 2, 1893, and those of the Manufactures and Liberal Arts Building of the World's Columbian Exposition in the same periodical for Sept. 1 and 8, 1892. The form of the latter is shown in Fig. 87, Art. 60. As indicated in Fig. 83, the horizontal thrust of roof arches is usually taken by a tie located below the level of the floor.

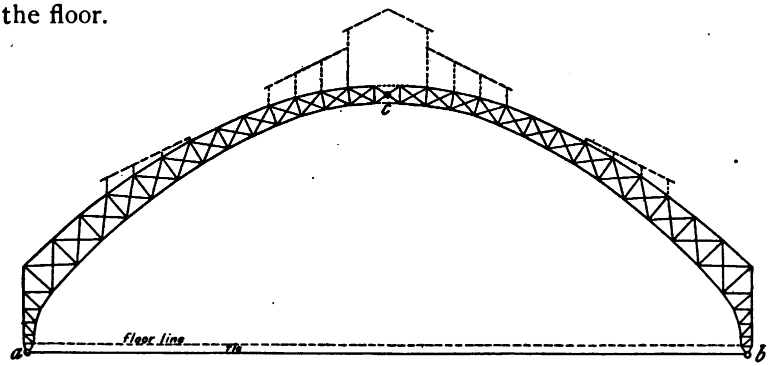

Fig. 83.

Three-hinged arches are analogous to simple trusses because their stresses are fully determinate statically, while those having two or less hinges are analogous to continuous structures.

Prob. 58. Refer to the Engineering News, June 30, 1892, and Nov. 9, 1893, and to the Engineering Record, April 20, 1889, March 21, 1891, and July 3, 1897, for descriptions of other metallic roof arches with three hinges. Copy their skeleton diagrams and record the principal dimensions.

ART. 60. REACTIONS OF THE SUPPORTS.

Let a single concentrated vertical load P be placed on the left half of an arch with three hinges, at a distance of kl from the left support as shown in Fig. 84. Let V_1 and V_2 be respectively the vertical components of the inclined reactions at the left and right supports, while H is their horizontal component. These quantities may be found by means of the three conditions of static equilibrium, viz.: The sum of the horizontal components of

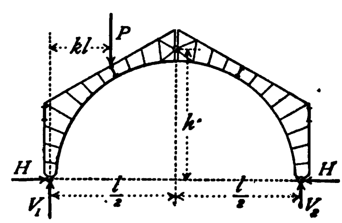

Fig. 84.

the external forces equals zero, the sum of their vertical components equals zero, and the sum of their moments equals zero. The first condition shows that the thrust H is the same at both supports. The second and third of the following equations are obtained by taking moments respectively about the right support and the hinge at the crown.

$$V_1 + V_2 - P = 0, \qquad V_1 l - P(l - kl) = 0,$$
$$\tfrac{1}{2} V_1 l - Hh - P(\tfrac{1}{2} l - kl) = 0,$$

whence $V_1 = P(1 - k)$, $V_2 = Pk$, and $H = Pkl/2h$. These values of V_1 and V_2 are exactly the same as if the load P were supported by a simple truss of the same span. The influence of the center hinge on the reactions therefore adds the horizontal thrust H. If the load P be placed on the right half of the arch and kl be measured from the left support the value of the thrust becomes

$$H = \frac{P(1 - k)l}{2h}.$$

Both expressions for H show that it varies inversely as the rise of the arch. In case a tie is employed, the magnitude of H equals that of the stress in the tie provided one end of the arch rests on rollers.

The vertical and horizontal components of the pressure of the right half against the left half of the arch in Fig. 84 must be equal to those of the reaction at the right support since there is no load on the right of the center hinge. The line of action of the inclined reaction R_2 at the right support must also pass through the middle hinge or there will be rotation of the right segment.

If the vertical load be replaced by an inclined or a horizontal one the reactions are found in a similar manner, by substituting the proper values of the lever arms of the load. The reactions due to any number of loads may be obtained by taking the sums respectively of the values of V_1, V_2 and H found for each apex load separately.

The reactions are readily found by graphics as indicated in Fig. 85. The line of action of R_2, which must pass through the hinges b and c as previously explained, is produced to meet the load at the point d, and then the line of action of R_1 is passed through the hinge a to the point d, since the three forces P, R_1 and R_2, being in equilibrium, must meet in a point. On drawing parallels to these lines through the extremities of P in the force diagram the values of R_1 and R_2 may be obtained by measurement with the same scale of force with which P is laid off. By drawing the horizontal through o, the values of H, V_1 and V_2 may be found, if desired. The two right lines mc and cn constitute the locus of the point of intersection d of any load and its corresponding reactions. These lines will be called the "reaction locus."

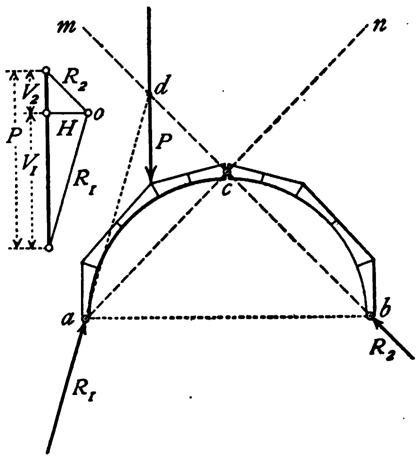

Fig. 85.

This method applies without any modification to horizontal and inclined forces or loads. In order to find the reactions due to any number of loads the resultant may be obtained for all the loads on each half of the arch and the respective reactions due to the two resultants may then be combined in the usual way. A neater and more expeditious method is given in Fig. 86. All the loads are laid off in succession on the load line, and by taking a pole o at random, two equilibrium polygons are drawn, one for each segment of the arch. On drawing rays parallel to their closing lines, the resultant of the loads on each segment is divided into two parts, thus giving the loads which are transferred to its supporting hinges by the action of the segment as a simple truss with inclined parallel reactions. The reactions of the arch are evidently not affected by replacing the

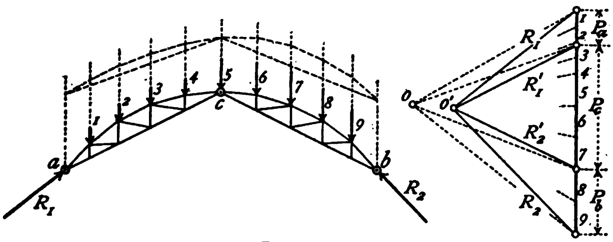

Fig. 86.

given loads by those just found. P_a, P_c and P_b are the loads transferred to a, c and b respectively. The load P_c acting alone at the crown causes the reactions R_1' and R_2' whose lines of action are ac and bc, and hence the rays marked R_1' and R_2' are drawn parallel to these lines. The loads P_a and P_b acting alone cause reactions at a and b respectively equal and opposite to these loads. The final reactions are therefore found by composition to be R_1 and R_2 as shown.

If some or all of the loads are inclined it may be preferable to take a separate pole for each equilibrium polygon to avoid polygons of inconvenient shape, size, or position. In Fig. 87 the construction is given to find the reaction due to the normal wind loads on the left side of the arch. The lines of action of P_a and P_c are parallel to the long chord of the load line which gives the direction and magnitude of the resultant of all the wind loads. The equilibrium polygon is drawn by means of the pole o. Only a few of the rays are shown. The reactions due to P_c are R_1' and R_2 drawn respectively parallel to ac and bc, and the final reactions are R_1 and R_2. A point in the line of action of the resultant is found by producing the first and last sides of the equilibrium polygon, and the results are checked by observing whether the reactions meet the resultant at the same point. In Fig. 87 the point falls beyond the limits of the diagram.

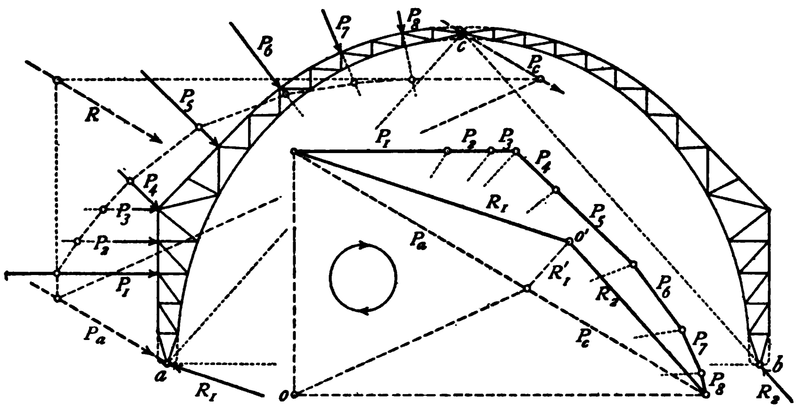

Fig. 87.

Prob. 59. The dimensions of the main arches of the Cleveland Arcade roof are given in the Engineering Record, March 21, 1891. Find the reactions due to a wind pressure of 40 pounds per vertical square foot on the roof, including the ventilator. The normal wind pressure is given in the table in Art. 19 of Part II.

ART. 61. STRESSES IN ROOF ARCHES.

After the reactions are obtained the stresses in the members of a three-hinged arch may be found either by the analytic or by the graphic method in the same way as for simple trusses. If the stresses are found analytically the method of moments is usually preferred to that of the resolution of forces (see Part I, Arts. 4, 5, and 7). In some cases, however, the latter method is more convenient for the web members after the chord stresses have been found by moments. While the analytic method is simple in theory its application to trussed roof arches often leads to tedious computations in finding the lever arms of the external forces and internal stresses. In all cases the results should be checked by measuring the lever arms on a truss diagram drawn to scale. If the diagram be carefully drawn to a large scale the lever arms may often be obtained with sufficient precision by measurement only.

In general the graphic method is the most convenient. Starting with the reaction R_1 in Fig. 86 (Art. 60) a stress diagram for the left segment may be drawn in the usual way. In order to avoid the accumulation of errors, it is desirable to work from each hinge toward the middle of the segment. This requires the pressure of the right segment against the left one to be known.

If the load 5 at the crown be regarded as on the right segment the pressure against the left segment at the hinge is represented by the ray joining o' with the point in the load line between the loads 4 and 5, the direction of the force being toward o' or toward the left. If the load 5 be regarded as on the left segment the required pressure is then represented by the ray drawn to the point between loads 5 and 6. An examination of the force diagram in Fig. 86 shows that if the vertical load on the two segments is symmetrical with respect to the vertical through the middle hinge both in magnitude and position that the reaction of one segment against the other is horizontal and equal to H, provided that one-half of the load at the hinge be regarded as on each segment.

It is frequently advisable also to check an intermediate stress. This may be done in the following manner. Let an equilibrium polygon be drawn whose first and last sides coincide with the lines of action of the reactions at a and c. This requires o' which is the intersection of the rays representing these reactions to be used as the pole. Fig. 88 is a portion of Fig. 86 after the equilibrium polygon is drawn. Let a section be passed cutting the middle panel of the segment. The side B of the equilibrium polygon, which lies between the lines of action of loads 2 and 3, is the line of action of the resultant of all the external forces on either side of the section. Considering the forces and the part of the truss on the left of the section the known direction of the resultant is indicated by the larger

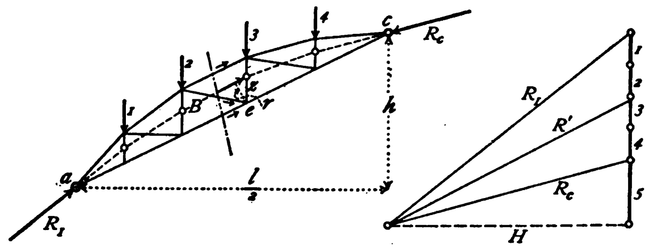

Fig. 88.

arrow, and the unknown stresses by the smaller arrows directed away from the section. The stress in the upper chord may now be found by taking moments about e, the intersection of the diagonal with the lower chord member cut by the section. The moment of the resultant is $R'r$ and when this is divided by the lever arm of the upper chord its stress is the quotient. In this case the stress is compression.

If the vertical distance z from e to the line of action of R' be measured, the moment $R'r$ may be replaced by Hz, for from similar triangles (similar, because the angles are equal), $r/z = H/R'$ whence $R'r = Hz$. Similarly the stress in the lower chord may be found by taking moments about the point where the diagonal intersects the upper chord. As the moment at the center hinge is zero the equilibrium polygon must pass through c.

If the dead load is symmetrical it requires a stress diagram for but one segment. The snow load also requires one diagram. For the wind load two diagrams are drawn for the same segment, since it requires less labor than to construct one diagram for each segment. Referring again to the left segment in Fig. 87 the first diagram is for the reactions R_1 and R_2 but with the latter applied at *c* instead of *b*, while the second is for two equal and opposite reactions applied at *a* and *c* respectively, whose magnitude equals R_2 and whose lines of action coincide with the chord *ac*. In all of these diagrams especial care must be exercised in drawing their lines parallel to the true direction of the truss members which are frequently much shorter than themselves. Where the panel points lie on arcs of circles, as is frequently the case, the direction of a chord member is best determined by means of the radius drawn to its middle point.

It is evident that to avoid ambiguity in stress only two members in either segment can meet at a hinge. Sometimes as shown in Fig. 83 members are added to give the appearance of a continuous curve at the crown for æsthetic reasons, but they are arranged so as to permit the necessary motion of the truss and do not transmit any stress. In other cases nearly the same effect is produced by the addition of members to each segment as illustrated in outline in Fig. 87. Let the student examine the details in the periodical to which reference is made in Art. 59. The treatment of roof trusses with counter-braces was fully illustrated by means of an example in Art. 63 of Part II.

Prob. 60. Find the stresses in the main arch whose skeleton diagram and dimensions are given in Engineering News, Nov. 9, 1893, due to a wind pressure of 40 pounds per vertical square foot.

ART. 62. METALLIC ARCHED BRIDGES.

The three-hinged arches used in bridges are usually composed of a horizontal upper chord and a curved or broken lower chord united by vertical and diagonal bracing as illustrated in Fig. 91, Art. 64. Such an arch is designated as "spandrel-braced." The hinge at the crown is generally placed in the lower chord, and the floor system is supported directly by the upper chord at the panel points.

The highway bridge across the Mississippi River at Minneapolis, erected in 1888, contains two arches with a span of 456 feet and a rise of 90 feet, these being the largest three-hinged arches in this country. The next in size is the Driving Park Avenue bridge at Rochester, N. Y., the span being 428 feet and the rise 67 feet. A description of this bridge may be found in the Engineering Record, July 18 and August 1, 1891. The Panther Hollow bridge in Schenley Park, Pittsburg, Pa., completed in 1898, has a single span of 360 feet and a rise of 45 feet. The arches are of the same type as the preceding one, and have 20 panels. Each of the arches of the Stony Creek bridge, on the Canadian Pacific Railroad, has two chords which are about 20 feet apart connected by vertical and diagonal bracing so as to divide the arch into 16 panels, while the weight of the floor and live load is transferred to the upper chord at only six points including its extremities. The span is 336 feet and the rise 80 feet $8\frac{13}{16}$ inches. An illustrated description and detail drawings are given in Engineering News, August 2, 1894. An arch of short span, but with a solid web, is described in Engineering News, August 16, 1890. The arch of the Brooklyn-Brighton Viaduct has its upper panel points located on three right lines while the lower panel points lie on a parabola. The details of this arch are given in Engineering News, October 25, 1894.

The Belle Isle Park bridge has a span of 48 feet $10\frac{5}{8}$ inches and a rise of 3 feet $3\frac{7}{8}$ inches. This arch is not only notable on account of its unusually small rise but also because considerable attention was paid to æsthetic considerations both in determining its form and in the addition of wrought iron ornament. It is briefly described and illustrated in the Engineering Record, May 13, 1893.

Prob. 61. Consult the engineering periodicals and make a list of all the three-hinged arched bridges whose principal dimensions can be found, giving for each arch the span, rise, and ratio of rise to span.

ART. 63. LIVE LOADS FOR MAXIMUM STRESSES.

To determine what panel points must be loaded with the live load to cause the greatest tension or compression in any member, it is necessary to investigate the influence of a single concentrated load in different positions. If for the truss shown in Fig. 89 the method of moments be used to find the stress in L_3, a section may be passed cutting U_4, D_4 and L_3, and the center of moments taken at the intersection of U_4 and D_4 which is the

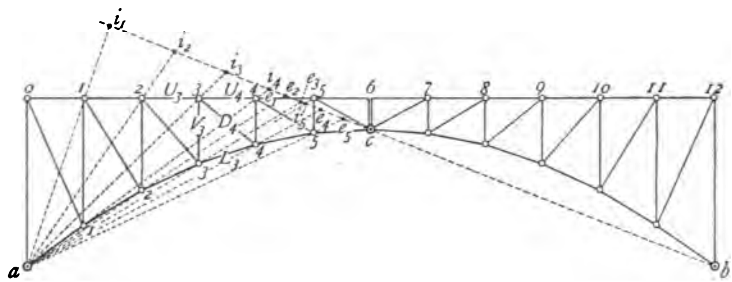

Fig. 89.

upper panel point 3. Let a line be drawn joining this panel point with the hinge a and extended to meet the reaction locus, which passes through the hinges b and c, at the point i_3. If any vertical load P be placed so that its line of action passes through i_3 it will produce no stress in L_3 because the lines of action of the reactions of the supports coincide with ai_3 and bi_3 as explained in Art. 60, and one of these passes through the center of moments of L_3. If P be placed anywhere on the right of this position it will cause compression in L_3, as may be readily seen on considering the truss on the left of the section, which is subject to only a single external force, namely, the reaction whose line of action now passes below or to the right of the center of moments. If, however, P be placed on the left of i_3 it will cause tension in L_3 as may be observed by considering the portion of the truss on the right of the section. The greatest tension due to the live load is therefore obtained when all the panel points on the left of i_3 are loaded, and the greatest compression when those on the right of i_3 are loaded.

The chord member L_5, adjacent to the center hinge, has its center of moments on the opposite side of the reaction locus from those of the rest of the chords and hence loads on the left of i_5 will also produce compression. The greatest compression in L_5 is therefore due to the live load covering the entire truss. If the upper chord were lowered so that the line bc passed through the upper panel point 5 then the load on its left would produce no stress in L_5 and in that case it would be immaterial whether the panel points on the left of 5 were loaded or not.

The required loading for nearly all of the chord members of this arch differs from that for simple trusses of the usual type because in simple trusses the chords cut by a section (which cuts only three members) intersect outside of the lines of action of the reactions, whereas in the three-hinged arch with spandrel bracing the form is generally such that for nearly all of its members the chords cut by a section intersect within the reaction lines.

The corresponding points which mark the positions where a concentrated load produces no stress in the different members of the upper chord, are marked e. All loads on the left of e_4 cause compression in U_4 while those on the right of e_4 cause tension in it. If the center hinge were placed in the upper chord, in which case the diagonals in the two adjacent panels would be reversed, the position of e_5 would fall below the hinge and then it would no longer serve as a point of division between the loads which produce tension and compression in U_5, but the greatest compression would be caused when the live load covered the whole truss.

In Fig. 90 the centers of moment for the web members are marked c_0, c_1, c_2, etc. These centers are joined with the hinge a, and the points where the connecting lines intersect the reaction locus are marked i_0, i_1, i_2, etc. The subscripts correspond to those of the lower chord members which are produced and hence also equal those of the verticals to which they apply, while they are one less than those of the corresponding diagonals.

When a load is placed with its line of action passing through i_3 it produces no stress in V_3, and hence i_3 marks a division between adjacent loads which produce stresses of opposite

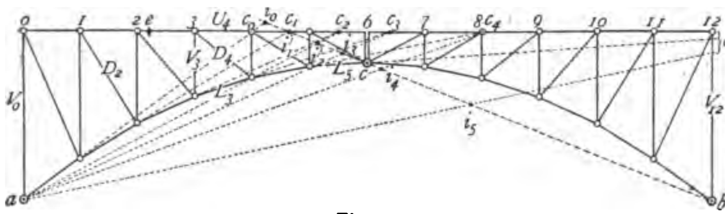

Fig. 90.

character in V_3. When, however, the point i falls below or on the right of the center hinge it ceases to have this significance because the load is now on the right half of the arch and the reaction of the right support no longer coincides with bc. If a load could be placed at i_4 and supported by the left half of the arch by means of an extension beyond the center hinge, then the two reactions would intersect at i_4, but with the usual construction the left reaction cannot pass below the center hinge.

When a center, as for instance c_3, lies on the right of the line bc there will also be another point of division for the live loads and which is located in the panel cut by the section through V_3, that is, between the panel points 2 and 3 of the upper chord, which in this case supports the floor. Let this point be marked e. This result is in accordance with the investigation in Art. 48 of Part II, which depends on the fact that the chords cut by the section intersect beyond the lines of action of the reactions. For a load on the left half of the arch, which may be regarded as a simple truss with two inclined reactions at a and c, the reaction at c acts in the line bc and hence if c_8 is beyond bc it satisfies the condition. The panel loads at 3, 4 and 5, which lie between the two points of division e and i_8, cause compression in V_8, while the remaining panel loads at 1, 2, 6, 7, 8, 9, 10 and 11 cause tension. The half panel loads at 0 and 12 need not be considered except for the stresses in the end verticals respectively, as they do not affect the stress in any other member. The greatest compression in V_0 is caused by panel loads 0, 1, 2, 3 and 4, and the greatest tension by panel loads 5 to 11 inclusive.

The center of moments for D_4 is c_3 and one point of division is at i_8 while another one is between panel points 3 and 4; hence the greatest tension is produced by the panel loads 4 and 5, and the greatest compression by the remaining loads. Since in Fig. 90 the point i_1 coincides with c_1, it shows that it is immaterial whether or not a point of division between 1 and 2 be employed for D_2. In other words the panel load at 1 does not produce any stress in D_2. In the same manner it may be shown that since i_4 and i_5 fall below the center hinge they cease to be points of division, which indicates that the reaction locus does not extend below the center hinge for vertical loads.

If a uniform live load be employed, the positions of the points of division, described above, will limit the loading for maximum and minimum stresses. This will give two or more partial panel loads for each position required. It is an unnecessary refinement to find the true limiting stresses due to a uniform load, and the use of equal panel loads for arches is preferred for the same reasons as for simple trusses.

Prob. 62. Prepare a table showing the loading for all the members of the arch in Fig. 91, provided the center hinge is placed in the upper chord and the diagonals in the two panels at the middle are reversed.

ART. 64. COMPUTATION OF STRESSES.

Let the arch be taken whose dimensions are given in Fig. 91, the dead panel load being 80 kips and the live panel load 24

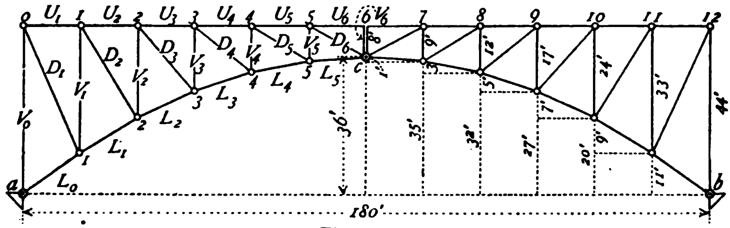

Fig. 91.

kips. The dead load is considerably greater than the live load on account of the heavy pavement of the roadway and sidewalks.

The vertical and horizontal components of the left reaction are computed as in Art. 60. The value of V_1 for P_{11} is $\frac{1}{12}P$, while $H = \frac{1}{12} \cdot \frac{180}{2 \times 36} P = \frac{5}{24} P$. The values of V_1 and H for the other panel loads are simple multiples of these values and for convenience in combining the results due to various positions of the live load the accompanying table is arranged. Since the half panel loads at 0 and 12 affect no members except the verticals directly below them it will save labor to omit them from the table, and finally to add a half panel load to the compression in each end vertical.

	V_1	H
P_1	$11 \times \frac{1}{12}P$	$1 \times \frac{5}{24}P$
P_2	10	2
P_3	9	3
P_4	8	4
P_5	7	5
P_6	6	6
P_7	5	5
P_8	4	4
P_9	3	3
P_{10}	2	2
P_{11}	1	1
Full load	$5.5P$	$7.5P$

In Art. 63 it was found that the greatest tension in the upper chord member U_4 is due to the live panel loads 6 to 11 inclusive. Therefore

$$V_1 = (6 + 5 + 4 + 3 + 2 + 1)\tfrac{24}{12} = 42 \text{ kips},$$

and $$H = (6 + 5 + 4 + 3 + 2 + 1)\frac{5 \times 24}{24} = 105 \text{ kips}.$$

Considering the forces on the left of the section cutting U_4, D_4 and L_3, and taking moments about the lower panel point 4,

$$42 \times 60 - 105 \times 32 + S \times 12 = 0, \text{ whence } S = +\ 70.0 \text{ kips}.$$

The greatest tension in U_4 is due to panel loads 1 to 5 inclusive, and for this loading $V_1 = 90$ kips and $H = 75$ kips. The equation of moments is

$$90 \times 60 - 75 \times 32 - 24(45 + 30 + 15) + S \times 12 = 0,$$

whence $S = -\ 70.0$ kips. These results indicate that under full live load the stress in U_4 is zero and hence the dead load stress is also zero. The above values are therefore the maximum and minimum stresses.

The greatest tension in the diagonal D_4 occurs under panel loads 4 and 5 and the corresponding values of V_1 and H are 30 and 45 kips respectively. The center of moments for D_4 is on the upper chord at a distance of 6 feet beyond the center hinge, as shown in Fig. 92. Its lever arm is 31.86 feet. Taking the moments of the forces on the left of the section,

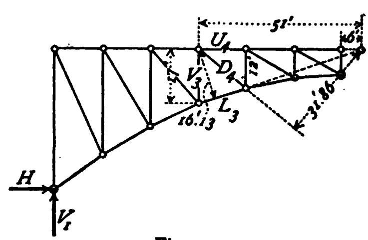

Fig. 92.

$$30 \times 96 - 45 \times 44 - S \times 31.86 = 0, \text{ whence } S = +\ 28.2 \text{ kips}.$$

When the load is at all the panel points except 4 and 5, $V_1 = 102$ and $H = 135$ kips. The equation of moments is then

$$102 \times 96 - 135 \times 44 - 24(81 + 66 + 51) - S \times 31.86 = 0,$$

which gives $S = -28.2$ kips. The dead load stress in D_4 is therefore zero.

In a similar manner the following reactions and live load stresses are obtained for the vertical V_8, the section in this case cutting U_8, V_8 and L_3:

LOADS.	V_1.	H.	S.
3 – 5	48	60	– 38.6 kips.
1, 2, 6 – 11	84	120	+ 14.6
1 – 11	132	180	– 24.0

Under full load the stress in V_8 is $-38.6 + 14.6 = -24$ kips, which being equal to the live panel load checks the result; for since under this load there is no stress in D_4, a section may be cut around the upper panel point 3, which shows that the stress in V_8 holds in equilibrium a single panel load. The dead load stress is consequently also equal to a panel load, or – 80 kips. The maximum stress is – 118.6 and the minimum stress is – 65.4 kips.

The reactions and live load stresses for the lower chord member L_8 are as follows:

LOADS.	V_1.	H.	S.
4 – 11	72	150	– 208.3
1 – 3	60	30	+ 18.6
1 – 11	132	180	– 189.7

As under full load there are no stresses in the diagonals the horizontal component of the stress in any lower chord member must be equal to H. Hence the stress in L_8 equals the product of H and the secant of the angle which L_8 makes with the horizontal. This gives a compression of $180 \times 1.0541 = 189.7$ kips, which equals the sum of the two stresses given above. As the dead panel load is ten-thirds of the live, the dead load stress in L_8 is – 632.4 kips. The maximum and minimum stresses are – 840.7 and –613.8 kips.

The vertical V_6 supports only the panel load at 6. If only a single post is used for V_6 one of the adjacent upper chords must be so arranged as to allow free movement due to the deflection of the truss and to changes in temperature. There is no stress in U_6 as it serves merely to hold V_6 in position. The stresses in the remaining members may be found in the same manner as those found above. As the arch is symmetrical the stresses in corresponding members of the two segments are equal.

Prob. 63. Compute the maximum and minimum stresses in the remaining members of the arch which was used in the preceding example.

ART. 65. PARABOLIC LOWER CHORD.

In the example in the preceding article it was found that the stresses in the diagonals and upper chord were zero under full load while the stress in each vertical equalled the panel load. This is due to the fact that the panel points of the lower chord lie upon a parabola whose vertex is at the center hinge. For, since the panel loads are equal, the equilibrium polygon for the arch must be a parabola and as there can be no moment at the hinges it must pass through them and will consequently also pass through every panel point of the lower chord. This indicates that there will be no bending moment at any lower panel point and hence no stresses in the upper chord; and since the horizontal component of the stress in any diagonal equals the difference in the stresses in the two chord members meeting its upper extremity there is therefore no stress in the diagonals. Under this loading then, the verticals simply transfer the equal panel loads to the lower panel points and the lower chord sustains all the load. This arrangement may be regarded as an inverted suspension system.

For three-hinged arches with parabolic lower chords, it is only necessary therefore to compute either the greatest tension or the greatest compression in all the members due to the live load together with the compression in the lower chord under full load. The least labor will be required if the greatest live load tension is computed for the upper chord and the diagonals, and the compression in the lower chord and verticals, as no panel loads are then on the left of the section except in a few cases.

If w be the load per linear unit and l the span, the reactions V_1 and V_2 each equal $\frac{1}{2}wl$, and H equals $wl^2/8h$. As the floor system transfers the uniform load to the panel points of the truss, these values of V_1 and V_2 include the half panel loads at the ends. The value of the reactions does not depend upon the form of the curve of the lower chord, and both V_1 and V_2 are also independent of the position of the center hinge. The value of H, however, depends upon the position of the center hinge which is generally placed at mid-span.

The lower chord may be given any curve of graceful form, but if it is not parabolic it requires the computation of an additional set of stresses. The lower chord of the arch at Thirtieth Street, near Market, in Philadelphia is elliptical or nearly so. See The Pennsylvania Railroad by JAMES DREDGE.

Prob. 64. Compute the maximum and minimum stresses in the lower chord of the arch in Fig. 91, provided its panel points are placed on the arc of a circle which passes through the hinges.

ART. 66. GRAPHIC ANALYSIS OF STRESSES.

If the lower chord of a three-hinged arch is not parabolic the dead load stresses are found by stress diagrams in the same manner as that described in Art. 61 for roof arches. If, however, the diagram be drawn for a uniform panel load of 1 kip the stresses due to the full live load as well as the dead load may be obtained by multiplying the stresses given by the diagram with the corresponding panel loads. When the lower chord is parabolic the dead load stresses may be found in a simpler manner like that for full live load to be explained later in this article.

In the graphic determination of the live load stress in any member it is desirable to obtain the stress in two parts: first, that due to the vertical component V_1 of the reaction of the left support; and second, that due to the corresponding horizontal component H of the reaction.

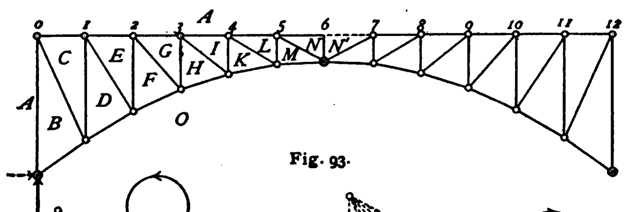

Fig. 93.

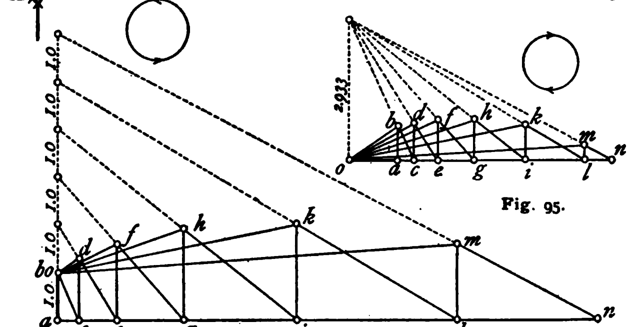

Fig. 95.

Fig. 94.

Let the dimensions and panel loads of the arch in Fig. 93 be the same as those given in Art. 64. Let a stress diagram be drawn for the left half of the truss under the assumption that this half is fixed at the right end and subject to a single vertical force of 1 kip at the left hinge. The stresses thus obtained are the same in all members on the left of the diagonal MN as that portion of the stresses due to V_1 if a concentrated load be placed at panel point 5 when the truss is supported in the ordinary way, provided the load is of such a weight as to cause V_1 to be equal to 1 kip. Of course, if any other unit than the kip be employed, the stresses will be expressed in terms of the same unit. Such a diagram is shown in Fig. 94. Next let a similar stress diagram be drawn when a horizontal thrust of 1 kip is applied at the left hinge. This diagram is given in Fig. 95.

The required position of the equal panel loads is found graphically as explained in Art. 63, and by the simple tabulation given in Art. 64 the corresponding values of V_1 and H are readily obtained. If now the stress in any member given by Fig. 94 be multiplied by the number of kips contained in the actual value of V_1 and that given by Fig. 95 be multiplied by the corresponding number for H, the algebraic sum of the products will be the required stress, provided, that for the given loading no panel loads are situated on the left of the section which would be passed if the analytic method of moments were adopted.

With diagrams drawn to the scale of one kip to an inch the stresses obtained for the lower chord member OH were $+2.79$ and -2.73 kips respectively. When the live panel loads 4 to 11 inclusive are on the truss $V_1 = 72$ kips and $H = 150$ kips. The stress in OH is therefore $S = +\ 2.79 \times 72 - 2.73 \times 150 = -\ 208.6$ kips. The computation in Art. 64 gave $-\ 208.3$ kips for the same member designated as L_3.

The following table gives the corresponding values expressed in kips for the remaining members whose stresses are given in Art. 64. S_v and S_h designate the stresses due to the vertical and horizontal forces of 1 kip applied at the left hinge respectively.

Member.	Loads.	V_1.	H.	S_v.	S_h.	S.
AI	6–11	42	105	− 5.00	+ 2.667	+ 70.0
GH	3–5	48	60	− 1.89	+ 0.865	− 38.8
HI	4–5	30	45	+ 3.02	− 1.38	+ 28.5

Since the values of S_v and S_h are to be multiplied by large quantities it is important that they be determined with care. In order to promote accuracy in the construction of the stress diagrams a diagram may be prepared like Fig. 96 whose inclined lines are respectively parallel to the diagonals and lower chords. A large scale is employed to lay off the horizontal and

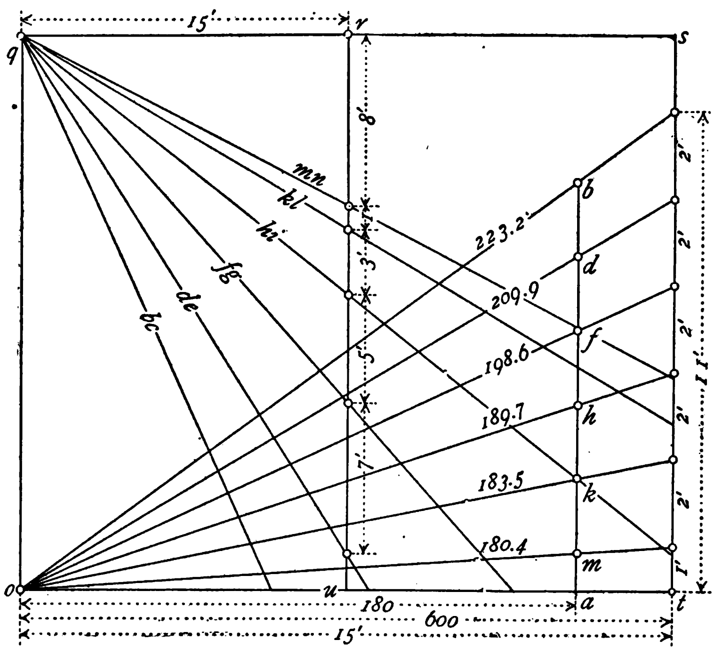

Fig. 96.

vertical components of the lengths of the members as shown, the scale for the chords being larger than that for the diagonals. Let the student compare the measurements on Fig. 96 with those of the truss given on Fig. 91. It is desirable that none of the lines in either stress diagram be longer than the corresponding parallel in this figure.

The diagrams also possess properties which afford valuable checks on their construction. In Fig. 94 the lines parallel to the diagonals cut off equal distances on the load line, and in Fig. 95 the corresponding lines meet at a point vertically above *o*, the intercept being in this case $44 \times 1/15 = 2.933$ kips. These quantities are the height of the upper chord above the end hinges, the value of *H*, and the panel length respectively. It is also desirable to compute the value of the largest stress in the upper chord. For instance, the stress in *AN* due to a vertical reaction of 1 kip is $90/8 = 11.25$ kips, while that for a thrust of 1 kip is $36/8 = 4.5$ kips.

The use of the circular arrows, in determining the character of the stresses, was fully explained in Art. 17 of Part II. It may be well, however, to give the following illustration. The external force *OA* in Fig. 93 was laid off on the load line in Fig. 94 from *o* toward *a*, the order of these letters indicating that the spaces *O* and *A* were taken with reference to the hinge, or to the truss, in the order indicated by the arrow. On taking the letters located around the lower panel point 3 in the same circular direction, they come in the order *O–H–G–F–O*. Following the direction indicated by the same order *o–h–g–f–o* on the stress diagram the first of the stresses acts from *o* toward *h* or toward the right. On transferring this direction to joint 3 the stress acts away from the joint and is therefore tension. Similarly the stresses *hg*, *gf* and *fo* are found to be compression, tension, and tension respectively.

The stresses in the lower chord under full load are obtained by laying off the horizontal *oa* in Fig. 96 equal to $H = 7.5 \times 24 = 180$ kips (see table in Art. 64), erecting the vertical *ab* and measuring the lines or rays *om* to *ob* inclusive. The values obtained are marked on the diagram and are expressed in kips.

The maximum and minimum stresses in the lower chord may now be computed. In the following table the stresses in the third line are found by subtracting those in the second from the stresses in the first line. The dead load stresses are $80/24 = 10/3$ times the live load stresses under full load. As this is a simple ratio they are computed, otherwise the dead load stresses would be found like those due to live load by laying off $H = 7.5 \times 80 = 600$ kips (using a different scale) and drawing another vertical in Fig. 96 as indicated.

	OB	*OD*	*OF*	*OH*	*OK*	*OM*
Full live load	−223.2	−209.9	−198.6	−189.7	−183.5	−180.4
Live load −	−223.2	−213.8	−209.9	−208.6	−200.3	−180.4
Live load +	0	+3.9	+11.3	+18.9	+16.8	0
Dead load	−744.0	−699.7	−662.0	−632.3	−611.7	−601.3
Maximum	−967.2	−913.5	−871.9	−840.9	−812.0	−781.7
Minimum	−744.0	−695.8	−650.7	−613.4	−584.9	−601.3

The stresses in the verticals are given in the following table:

	AB	*CD*	*EF*	*GH*	*IK*	*LM*	*NN'*
Full live load	−12.0	−24.0	−24.0	−24.0	−24.0	−24.0	−24.0
Live load −	−51.3	−56.5	−47.2	−38.8	−33.8	−43.9	−24.0
Live load +	+39.3	+32.5	+23.2	+14.8	+9.8	+19.9	0
Dead load	−40.0	−80.0	−80.0	−80.0	−80.0	−80.0	−80.0
Maximum	−91.3	−136.5	−127.2	−118.8	−113.8	−123.9	−104.0
Minimum	−0.7	−47.5	−56.8	−65.2	−70.2	−60.1	−80.0

The maximum and minimum stresses in the upper chord, beginning at the end, are: ± 17.9, ± 38.3, ± 58.5, ± 70.0 and ± 58.4 kips; and those in the diagonals, beginning at the left, are: ± 43.4, ± 38.2, ± 32.6, ± 28.5, ± 32.0 and ± 65.5 kips.

On comparing these results with the computed values, the lever arms being expressed to the nearest hundredth of a foot, the greatest difference in the maximum stresses of any lower chord member was found to be 0.1 kip, in the minimum 0.5 kip, and in any vertical 0.8 kip, the differences with but few exceptions not exceeding 0.2 kip. Although these differences are practically insignificant they might have been reduced if a larger scale than 1 kip to the inch had been used for the original of Fig. 95. The scale employed was smaller than should be used in practice.

The design of this arch, after making due provision for the reversal of stresses in the upper chord and diagonals, will show that about one-half of the material in the truss is distributed in the lower chord. It will also be noticed on examining the stresses that a number of sections can be made alike, thus reducing the cost of construction.

Prob. 65. Refer to Engineering News, Oct. 25, 1894, for the form and dimensions of the three-hinged arch of the Brooklyn-Brighton Viaduct, and find the stresses in all the members of the truss for a live load of 2 kips per linear foot.

ART. 67. POSITION OF WHEEL LOADS FOR CHORDS.

Let it be required to determine the position of a locomotive and train which shall cause the live load stress in any chord

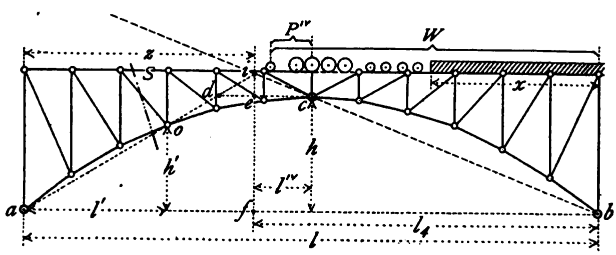

Fig. 97.

member when the positive bending moment is a maximum. Let S be the stress in the chord member cut by the section indicated in Fig. 97 and whose center of moments is at o. Using the same general notation as in Art. 49 of Part. II, W is the weight of the locomotive and train, and g the distance of its center of gravity from the right support b; P^{iv} the part of the load which is on the left of the center hinge c, and g^{iv} the distance of its center of gravity from c.

Since V_1 has the same value as if the arch were a simple truss (Art. 60),

$$V_1 = \frac{Wg}{l}.$$

Taking moments about the center hinge,

$$\tfrac{1}{2} V_1 l - Hh - P^{iv} g^{iv} = 0.$$

Substituting the value of V_1 and reducing,

$$H = \frac{Wg}{2h} - \frac{P^{iv} g^{iv}}{h}.$$

Remembering that no load can be on the left of the center of moments, as shown in Art. 63, the equation for the bending moment at o may now be written,

$$M = + V_1 l' - Hh'.$$

Substituting the values of V_1 and H, and reducing,

$$M = \left(\frac{l'}{l} - \frac{h'}{2h}\right) Wg + \frac{h'}{h} P^{iv} g^{iv}.$$

If the train advances a distance dx, both g and g^{iv} receive an increment equal to dx, and the bending moment receives an increment of

$$dM = \left(\frac{l'}{l} - \frac{h'}{2h}\right) W dx + \frac{h'}{h} P^{iv} dx.$$

Placing the derivative equal to zero gives the condition which makes M a maximum, which is

$$P^{iv} = \left(\frac{1}{2} - \frac{hl'}{h'l}\right) W.$$

Let this be transformed into the following:

$$P^{iv} = \left(\frac{l}{2} - \frac{hl'}{h'}\right) W/l.$$

On referring to Fig. 97 it is seen that the coefficient of W is the horizontal distance dc, and on account of similar triangles, this distance is to the span l as the distance $ec = l^{iv}$ is to $fb = l_4$. Substituting the latter ratio for the former, there follows,

$$P^{iv} = \frac{l^{iv}}{l_4} W.$$

This formula is similar in form to that deduced in Art. 49, Part II, for the maximum stress in the web members of simple trusses with broken chords, and is therefore applied in the same manner. To satisfy this criterion for loading a wheel must always be placed at the center hinge, and while generally the first wheel is at the right of i, it may sometimes happen that the condition is satisfied when it is a little to the left of i.

If it be desired to determine the position of the point of division by computation it may be done by means of a formula deduced as follows: The equations of the straight lines aoi and bci are respectively,

$$y = \frac{h'}{l'} z, \text{ and } y = 2h - \frac{2h}{l} z,$$

y being the ordinate corresponding to the variable abscissa z. At the point of intersection i, the two ordinates are equal, and on equating these values and reducing,

$$z = \frac{l}{1 + \frac{h'}{2l'} \cdot \frac{l}{h}}.$$

For example, let the dimensions of Fig. 97 be the same as those of Fig. 91, then the value of z indicated in the diagram is

$$z = \frac{180}{1 + \frac{27}{2 \times 45} \cdot \frac{180}{36}} = 72 \text{ feet.}$$

The load must therefore be so placed that (90–72)/(180–72) or one-sixth of the whole load on the truss is on the left of the center hinge. Any portion of the weight of the wheel placed at the hinge may be regarded as being either on the left or on the right of the hinge.

The maximum negative moment at o is produced when the live load comes on the bridge from the left. On account of the arrangement of live load diagrams it is more convenient to find

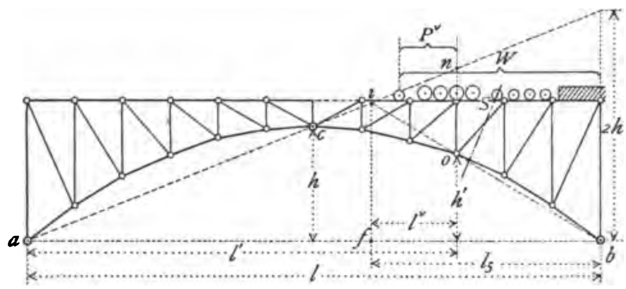

Fig. 98.

the maximum positive moment at the corresponding center of moments in the right half of the arch as indicated in Fig. 98. The reactions are

$$V_1 = \frac{Wg}{l}, \text{ and } H = \frac{Wg}{2h},$$

and if g^v is the distance of the center of gravity of the load P^v from the vertical through the center of moments, the bending moment is

$$M = \left(\frac{l'}{l} - \frac{h'}{2h}\right) Wg - P^v g^v.$$

When the load advances dx, both g and g^v receive the increment dx, and M the increment

$$dM = \left(\frac{l'}{l} - \frac{h'}{2h}\right) Wdx - P^v dx.$$

Placing the derivative equal to zero, the condition is found which makes M a maximum,

$$P^{\mathrm{v}} = \left(\frac{l''}{l} - \frac{h'}{2h}\right) W.$$

This may be written in the following form:

$$P^{\mathrm{v}} = \frac{\dfrac{2hl''}{l} - h'}{2h} W,$$

in which the numerator of the fraction equals the vertical intercept on at the center of moments between the line boi and the line aci produced. From similar triangles, the

$$\text{intercept } on = 2h \frac{l^{\mathrm{v}}}{l_5}.$$

Substituting this value, the above equation reduces to its final form

$$P^{\mathrm{v}} = \frac{l^{\mathrm{v}}}{l_5} W.$$

This formula is similar to the one for maximum positive moment in the left half of the truss, the center of moments in this case taking the place in a measure of the center hinge in the previous one. To satisfy the criterion a wheel must always be placed over the center of moments and in general the first wheel is to the right of i although it may sometimes advance a little beyond i.

The two formulas deduced in this article for the position of the wheel loads apply also to the lower chord whether the bracing contains verticals or not. In case the bracing is all inclined, a new formula has to be deduced for the upper chord which contains the loads Q like the corresponding formulas in Art. 61 of Part I, or in Art. 56 of Part II, deduced for simple trusses. Such a formula will also be affected by the character of the bracing at the center hinge, the simpler form being obtained either when a vertical is above the hinge, or the hinge itself is in the upper or loaded chord. As the use of the load line was fully illustrated in several examples in Chaps. IV, V and VI of Part II, it appears unnecessary to add another example.

Prob. 66. Find the positions of WADDELL'S compromise standard, Class U, for the chords in Fig. 91. This load has the spacing given in Fig. 24, Art. 17, the weight on the pilot wheel being 20 000, on each driver 40 000, and on each tender wheel 23 000, while the uniform train load is 4 000 pounds per linear foot.

ART. 68. POSITION OF WHEEL LOADS FOR WEB MEMBERS.

There are two points of division where a concentrated load may be placed without producing any stress in the diagonal *S*

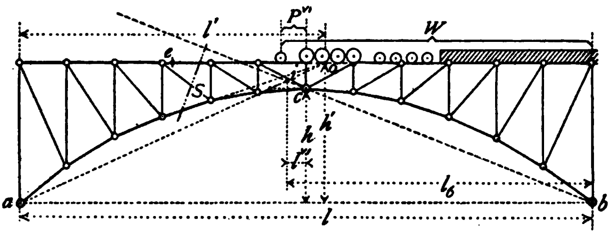

Fig. 99.

in Fig. 99, as explained in Art. 63, one of these being at *e* and the other in the vertical through *i*. The greatest compression is produced in the diagonal when one train, advancing from the right, covers the truss on the right of *i*, and another train, advancing from the left, covers the truss on the left of *e*. Let the position of each of these trains be considered separately.

In Fig. 99 only the first of these trains is shown on the truss; and it is required to find its position so as to make the moment of all the external forces on the left of the section, with respect to the center of moments *o*, a maximum. The center *o* is at the intersection of the chords cut by the section indicated. On comparing the diagram with Fig. 97 it is observed that the conditions are similar, and hence by the same reasoning the following formula is deduced:

$$P^{vi} = \frac{l^{vi}}{l_6} W.$$

This condition may be satisfied by placing a wheel directly over the center hinge and the pilot wheel near the vertical through *i*.

To find the required position of the second train let the truss be reversed as in Fig. 100 in order to bring the train on from

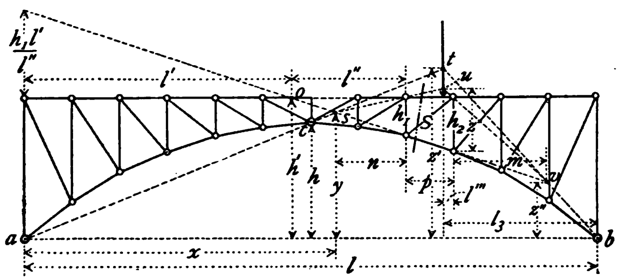

Fig. 100.

the right. By the same reasoning as in Art. 48 of Part II, the arrow is located and gives the position of a concentrated load which causes no stress in the diagonal *S*. This point is the same as *e* in Fig. 99. The right half of the arch is subject to two inclined reactions whose lines of action are *act* and *bt* respectively. The sides *su* and *vu* of the equilibrium polygon *suv* pass through the panel points of the loaded chord which are next to the given section, and its vertices *s*, *v* and *u* are on the lines of action of the reactions and of the concentrated load. As the three points *t*, *u* and *v* depend upon the position of the concentrated load, its position must be found by trial.

Now let the locomotive and train cover the truss on the right of the arrow, their position being required which shall make the moment of the forces on the left of the section, with respect to the center o, a maximum. The wheels are omitted in Fig. 100 to avoid confusion of lines. Let P''' be the load on the panel, and g''' the distance of its center of gravity from the right end of the panel, W the total load and g the distance of its center of gravity from the right support. Proceeding in the same manner as before,

$$V_1 = \frac{Wg}{l}, \qquad H = \frac{V_1 l}{2} / h = \frac{Wg}{2h},$$

and
$$M = \left(\frac{l'}{l} - \frac{h'}{2h}\right) Wg + \frac{l''}{p} P'''g'''.$$

When the load advances dx both g and g''' receive the increment dx, and M the increment

$$dM = \left(\frac{l'}{l} - \frac{h'}{2h}\right) Wdx + \frac{l''}{p} P'''dx.$$

On equating the derivative to zero, there is found

$$P''' = \frac{p}{l''}\left(\frac{h'}{2h} - \frac{l'}{l}\right) W.$$

On finding successively the values of y, x, n, z, m, z'', z' and h_1 (two values being obtained for z and also for z') in terms of the quantities which constitute the coefficient of W in the last equation, and combining these values as indicated by their relation in Fig. 100 there may be obtained after rather tedious reductions the equation,

$$\frac{p}{l''}\left(\frac{h'}{2h} - \frac{l'}{l}\right) = \frac{l'''}{l_3}.$$

Substituting this in the preceding equation the required criterion is found,

$$P''' = \frac{l'''}{l_3} W,$$

which is applied exactly like that in Art. 49 of Part II, or like the previous ones deduced in Arts. 67 and 68.

The greatest tension in the diagonal is caused when the wheel loads cover only the portion between e and i in Fig. 101, and it may be shown that the bending moment is

$$M=\frac{l'}{l}Wg-\frac{h'}{2h}W(l-g)-\frac{l''+p}{p}\cdot P^{\text{vii}}g^{\text{vii}},$$

while the loads must be so placed as to satisfy the condition expressed by the formula

$$P^{\text{vii}}=\frac{l^{\text{vii}}}{l_7}W.$$

In applying this formula it may frequently be found that no locomotive specified will cover the distance from e to i without

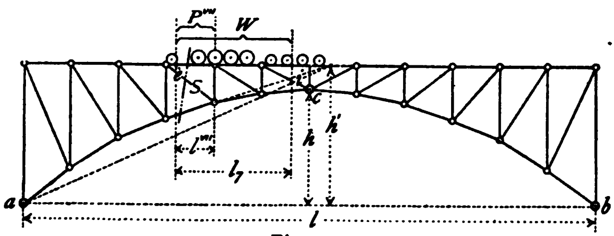

Fig. 101.

one or more wheels extending beyond these limits, but as there may be shorter locomotives in actual use which will produce a greater tension in the diagonal than the larger locomotive specified, it will be on the safe side to consider only those wheels of the specified locomotive which lie between the given limits. A driver is always placed at the panel point on the right of the section, and as much of the load as possible between e and i, while at the same time the preceding criterion is satisfied.

When, as stated in Art. 63, the point i falls below the center hinge it is no longer a point of division, and in that case the load must extend from e to the right support for the greatest tension in S. The criterion for position will not be of as simple a form as those already deduced, but if the pilot wheel is placed near e (usually on the right of it) and a driver at the panel point on the right of the section the position will generally be found correct. If there is any doubt between two positions the stresses due to both must be found and compared.

While the above investigation relating to web stresses has been made by using a diagonal, the same criteria apply also to the verticals, after the corresponding points of division e (Figs. 99 and 101) are located. It will be observed that all positions of the wheel loads for both chord and web members with but very few exceptions may be found graphically by means of the simple operation of stretching a thread when a tracing of the truss diagram is placed on the sheet containing the load line.

Prob. 67. On a truss diagram drawn to a scale of 10 feet to an inch, find the position of the point of division e for each diagonal, and on a second diagram determine the corresponding points of division for the verticals.

Prob. 68. Find the positions of the given load for the web members of the arch in Prob. 66.

ART. 69. STRESSES DUE TO WHEEL LOADS.

The equations for the bending moment in Art. 67 include the expressions Wg, $P^{iv}g^{iv}$, and $P^{v}g^{v}$, which may be read directly from the equilibrium polygon or moment diagram as indicated in Arts. 47 and 51, Part II. If the coefficients l'/l and $h'/2h$ are simple ratios their product with the quantities read from the moment diagram is better obtained analytically, but if not, it may save labor to use graphical arithmetic (Art. 14 of Part II), the necessary construction being made directly on the tracing paper containing the truss diagram. Since the loading is different for the two chords whose centers of moments lie in the same vertical, it is preferable to divide the bending moment for each chord member directly by its lever arm.

When the centers of moments of the web members lie within the limits of a drawing of convenient size, as is generally the case with spandrel-braced arches, it may be desirable to find the lever arms and divide the corresponding bending moments by them. Where the centers of moments are not conveniently located it is better to adopt the method developed in Art. 50 of Part II, and illustrated by examples in Arts. 52 and 53 of Part II. Its application to arches is exactly the same as to simple trusses after the moments are obtained for the panel points at the extremities of each diagonal. In view of the examples referred to the student should have no difficulty in writing an equation for the bending moment at any panel point of the arch, and in arranging its terms so as to facilitate its use.

When the load occupies a position like that in Fig. 101 it is required to find the moments at panel points, at the center hinge, or at the right support, when some wheel loads are considered not to be on the bridge. The properties of the moment diagram as indicated in Art. 42 of Part II, are so simple that any of these moments may be quickly found by producing one or more sides of the polygon to intercept the required value on the ordinate through the center of moments.

As examples of simple trusses were worked out in detail in Chaps. IV, V, and VI of Part II, illustrating the use of the moment diagram and the construction of the force polygons for the resolution of the shear no additional example will be given.

Prob. 69. Find the greatest tension and compression in the chords and diagonal of the fourth panel of the arch in Probs. 66 and 68, and also in the vertical on each side of this panel.

ART. 70. EXCESS LOADS.

If one excess panel load be employed in combination with given live panel loads, its position is the same as that indicated in Arts. 67 and 68 as the proper one for a driver of the typical consolidation locomotive in order that the respective criteria may be satisfied. These positions are the following: Above the hinge *c* in Fig. 97; above the center of moments *o* in Fig. 98; above the hinge *c* in Fig. 99; and at the right end of the panel cut by the section in Figs. 100 and 101.

When two excess panel loads are specified the first one is placed in the positions just stated, and the second one on the right of the first, provided the live load advances on the truss from the right as shown in Figs. 97 to 101 inclusive. As Figs. 98 and 100, however, give the loading for members in the right half of the arch, the loading for the corresponding members in the left half will evidently be symmetrical with that given in each diagram.

Prob. 70. Find the stresses due to an excess load of 12 kips in the members *OH*, *AI*, *GH*, and *HI* in the arch in Fig. 93, Art. 66.

ART. 71. WIND STRESSES.

The upper lateral system of a three-hinged arch whose center hinge is in the lower chord, as in Fig. 102, is not continuous on account of the provision made near *C* for motion due to deflection under live load and changes in temperature. The lower lateral system is continuous in this case. Sway bracing is provided between the verticals of the opposite trusses.

Let it be required to find the stresses in the lower lateral system and in the arch in Fig. 102 due to the wind pressure on both arches of the bridge, which are 20 feet apart, under the assumption that the pressure applied at the panel points of the upper chord is transmitted by the sway bracing to the lower chord and thence by the lower lateral system to the abutments. The area presented by each arch to the wind increases from the middle toward the ends, and strictly the wind panel loads should increase in the same manner. For the sake of simplicity, however, in the following computations let the wind load be taken at the usual value of 150 pounds per linear foot for each lateral system, and equally divided between the panel points of each chord. As the panels are 15 feet long the panel load for each chord of both systems is $75 \times 15/1000 = 1.125$ kips. Let the panel points of the leeward truss be designated by the primed or accented letters corresponding to those in Fig. 102.

The panel loads at C and C' produce an overturning moment with respect to c or c', through the sway bracing, equal to $(1.125 + 1.125)8 = 18$ kip-feet, which causes two equal vertical reactions at the latter points, downward at c and upward at c'. The trusses being 20 feet apart, these reactions are $18/20 = 0.9$ kip. The sum of the horizontal reactions at c and c' must also equal the sum of the panel loads at C and C', and these

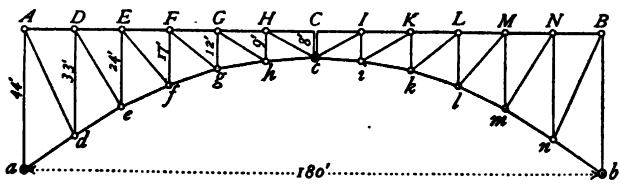

Fig. 102.

may be assumed to be equal, provided the sway bracing is stiff. If its diagonals would take only tension, the entire horizontal reaction would be applied at the windward point c. Because of the action of the sway bracing, the given horizontal loads at C and C' may therefore be replaced by equal loads at c and c', together with vertical loads of 0.9 kip applied at C and C', acting upward at C and downward at C'. The total horizontal wind load at c is $1.125 + 1.125 = 2.25$ kips, while that at c' and at all other panel points of the lower lateral system has the same value.

The stresses in the lower lateral system alone may hence be found by developing it into a horizontal plane, and treating it as a simple truss with parallel chords. Its span equals the actual length of the lower chord of the arch, and a load of 2.25 kips is applied at every panel point, except at the ends.

The equivalent vertical loads due to the overturning moments transmitted by the sway bracing are found to be the following: At C and C', 0.9; at H and H', 1.01; at G and G', 1.35; at F and F', 1.91; at E and E', 2.70; at D and D', 3.71; at A and A', 2.45 kips. In addition to these loads which cause stresses in the arches there must be found the equivalent vertical panel loads due to the overturning moments of the horizontal loads of 2.25 kips applied at both windward and leeward panel points of the lower lateral system.

Let the two panels $hcii'c'h'$ in the middle of the lower lateral system be considered separately and as supported at h, h', i and i'. The horizontal loads of 2.25 kips at c and c' will cause an overturning moment for this portion of the system equal to $(2.25 + 2.25)(9 - 8) = 4.5$ kip-feet. This must equal the sum of the moments of the equal and opposite vertical reactions at h and h' and at i and i', the lever arm of each couple being 20 feet. The same reactions would be caused by vertical loads of $4.5/20 = 0.23$ kip at c and c', acting upward at c and downward at c'. These two panels of laterals also transfer one half of the horizontal loads at c and c' to h and h' and the other half to i and i'. Hence so far as the arches are concerned the same stresses will be produced if the horizontal loads of 2.25 kips at c and c' are replaced by horizontal loads of 1.125 kips applied at h, h', i and i', respectively, together with an upward load of 0.23 kip at c and an equal downward load at c'.

The four panels extending from gg' to kk' are next to be considered as supported at g, g', k' and k, and subject to the horizontal wind loads of $2.25 + 1.125 = 3.375$ kips at each of the points h, h', i' and i. As the vertical loads referred to in the preceding paragraph do not cause any overturning moment they need not to be taken into account until later. The overturning moment for the left half only of this portion of the lateral system is $(3.375+3.375)(12-9)=20.25$ kip-feet. These loads may therefore be replaced by horizontal loads of 3.375 kips applied at g and g', together with a vertical upward load of $20.25/20=1.01$ kips at h, and an equal downward load at h'. The same change in loading is made for the right half. Similarly the horizontal loads of $3.375+2.25=5.625$ kips at g and g' may be replaced by equal horizontal loads at f and f' and by an upward load at g of $(5.625+5.625)(17-12)/20=2.81$ kips, and an equal downward one at g'. The equivalent vertical loads at f and f' are 5.51; at e and e', 9.11; at d and d', 13.61 kips.

When the truss in Fig. 102 is changed to the leeward one its stresses due to the total overturning moment of the wind pressure on the arches are equal to the stresses produced by all of the above equivalent vertical loads, acting downward. Since these stresses always occur in conjunction with the dead load stresses, they have the same magnitude in the windward as in the leeward truss while their signs are reversed. For convenience all the equivalent vertical loads may be applied at the upper panel points, the stresses in the verticals being corrected afterward by adding algebraically the lower panel loads.

For railroad bridges the equivalent vertical loads due to the pressure of the wind on the train are found in the same manner as for that on the truss. They are applied at the panel points of the upper chord and treated as a moving load. Under the assumption that the wind pressure on the train is transferred to the lower chord by the sway bracing, the lever arm of any horizontal wind panel load equals the distance of the center of pressure above the corresponding lower panel point. If, on the other hand, the pressure be regarded as transferred by the upper lateral system to the points A and A' and to C and C', the lever arm is the distance of the center of pressure above the plane of the lateral system. The actual stresses probably lie between the limiting values thus found.

Prob. 71. Find the horizontal wind panel loads to be used in determining the stresses in the lower lateral system, and the equivalent vertical loads on the arch in Fig. 102, under the assumption that each half of the upper lateral system transfers the wind loads to its extremities.

ART. 72. ARCH RIB WITH SOLID WEB.

One-half of a three-hinged arch with a solid web like that of a plate girder is typified in Fig. 103. The floor system is represented by the horizontal line while the verticals indicate the struts which support the floor and transmit the loads to the arch rib. If the outer parallel curves pass through the centers of gravity of the upper and lower flanges of the arch the maximum and minimum flange stresses at *n* may be obtained by finding the maximum and minimum bending moments with reference to

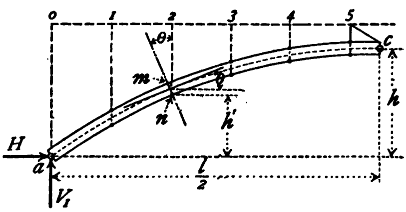

Fig. 103.

the center *m*, directly opposite *n*, provided the usual specification is made that the flanges shall take the entire bending moment while the web takes all the shear. The position of the live load and the values of the reactions and bending moments may be found in exactly the same manner as described in Arts. 63 and 64.

Sometimes the bending moment may be found more readily by observing the relation which it bears to its value when the arch is replaced by a simple truss. Since it was shown in Art. 60 that for vertical loads the vertical components V_1 and V_2 of the reactions of the arch are equal to those for a simple truss of the same span it is clear that the difference in the bending moment in the two cases is due simply to the moment of H. If, for example, it be required to find the bending moment M with reference to the center n in Fig. 103 whose height is h' above the horizontal through the hinges a and b, and if M' denotes the corresponding bending moment for the simple truss, then

$$M = M' - Hh'.$$

Similarly for the center hinge c the bending moment is

$$M_c = M'_c - Hh,$$

in which M'_c denotes the bending moment at c when the arch is replaced by a simple truss. But because the moment M_c at the hinge is zero, $H = M'_c/h$, whence

$$M = M' - \frac{h'}{h} M'_c.$$

In the application of this formula care must be exercised in certain cases in determining whether a given load is to be taken on one or the other side of the inclined section. For example, if the line mn, in Fig. 103, is drawn normal to the axis of the arch through its intersection by the vertical strut at 2, the load transmitted by the strut should be regarded on the right of the section for the flange stress at m, and on the left of the section for the flange stress at n, although the corresponding centers of moments are on opposite sides of the strut. If the centers of moments be taken in the same vertical section as indicated at the other struts in the diagram the load on the strut at each section does not need to be considered.

In order to find the shear in the section mn which is normal to the axis of the arch rib at that point, and makes an angle of θ with the vertical, let all the external forces on the left of the section be projected on the section. If V_1 and H are the vertical and horizontal components of the reaction at a, and ΣP the loads on the left of the section the shear in the given section is

$$V_n = (V_1 - \Sigma P)\cos\theta - H\sin\theta.$$

Since the expression $V_1 - \Sigma P$ equals the vertical shear at the section when the entire arch is treated as a simple beam, it may be denoted by V' and then the equation becomes

$$V_n = V'\cos\theta - H\sin\theta.$$

The position of the live load which makes V_n a maximum or a minimum may be found in the manner described in Art. 63 provided the shear be regarded as the stress in a web member whose center of moments is at infinity on the tangent to the axis of the arch rib at the given section. The shear due to the dead load may be most conveniently found by drawing the special equilibrium polygon which passes through the three hinges as shown in Fig. 88, Art. 61, and then projecting on the section the ray which is parallel to the side of the equilibrium polygon which is cut by the section. If the axis of the rib be parabolic and the dead panel loads are all equal the vertices of the equilibrium polygon will coincide with the points where the posts intersect the axis.

Prob. 72. The arch rib in Fig. 103 has a span of 84′ 4″, and a rise of 14′ 3″. Its axis is circular. In sections at one-third and two-thirds of the distance from the springing to the crown the effective depths of the rib are 20 and 18½ inches respectively. Find the maximum and minimum flange stresses and shears in these sections due to a live load of 12.3 kips per panel.

ART. 73. DEFLECTION.

The skeleton truss diagram in Fig. 104 represents the western arch of the Fairmount Park bridge at Philadelphia. The roadway is on a grade of 1.2 per cent and the panel points of the lower chord lie on a parabola whose vertex is at the center hinge c. The tangent to the parabola at c is horizontal. For the purpose of illustrating the method, the deflection of the

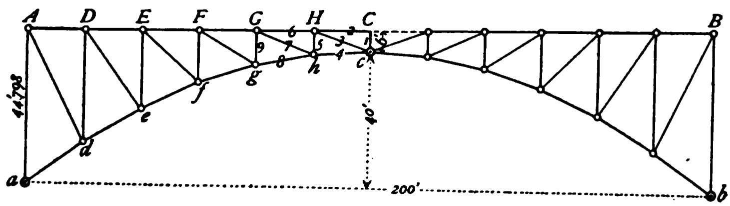

Fig. 104.

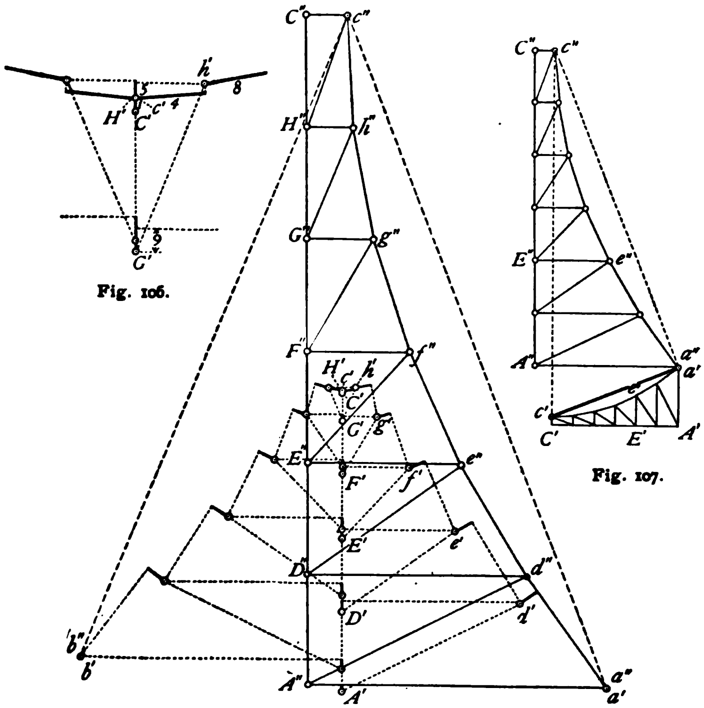

Fig. 106.

Fig. 105.

Fig. 107.

above three-hinged arch, due to a live panel load of 40 670 pounds, will be found. The value of H for a full live load is $7.5 \times 40\,670 = 305\,000$ pounds, and by means of a diagram similar to a portion of Fig. 96, the stresses in the lower chord are found, and inserted in the following table. There are no stresses in any other members except the verticals, the compression in the end verticals being a half panel load and in the others a full panel load.

The areas of cross-section of any two verticals, occupying symmetrical positions in the truss, are the same, and the lengths of the verticals in the right half of the arch are respectively 87.7, 130.1, 199.2, 294.9, 417.3 and 566.4 inches. The corresponding values of the deformation λ are 0.0103, 0.0152, 0.0233, 0.0328, 0.0353 and 0.0207 inches. The lower chord has the same cross-section throughout. The coefficient of elasticity is taken at 29 000 000 pounds per square inch, the material being medium steel. The principal dimensions of the arch and the lengths and section areas of the members were furnished by JOHN STERLING DEANS, Chief Engineer of the Phœnix Bridge Company.

MEMBER.	STRESS.	LENGTH.	CROSS-SECTION.	λ	MEMBER.
	Pounds.	Inches.	Square Inches.	Inches.	Number.
ad	− 378 000	247.6	47.6	− 0 0678	24
de	− 355 000	233.0	47.6	− 0.0599	20
ef	− 336 000	220.5	47.6	− 0.0537	16
fg	− 321 000	210.7	47.6	− 0.0490	12
gh	− 311 000	203.8	47.6	− 0.0459	8
hc	− 306 000	200.4	47.6	− 0.0444	4
Aa	− 20 340	537.6	19.2	− 0.0196	25
Dd	− 40 670	393.3	16.6	− 0.0332	21
Ee	− 40 670	275.7	12.6	− 0.0307	17
Ff	− 40 670	184.9	12.0	− 0.0216	13
Gg	− 40 670	120.5	12.0	− 0.0141	9
Hh	− 40 670	82.9	12.0	− 0.0097	5
Cc	− 40 670	72.0	12.0	− 0.0084	1

The displacement diagram in Fig. 105 is constructed by the method given in Arts. 66 and 67 of Part II, by assuming *C* and the direction of Cc as fixed. The values of λ are laid off on the diagram in the order indicated by the numerals on the truss diagram. Fig. 106 is an enlargement of that part of Fig. 105 which is drawn first. The value of λ_1 is laid off upward from C' to c'. In locating H' the values of λ_2 and λ_3 are both zero. The perpendicular drawn at the end of λ_2 and whose direction is perpendicular to the chord member CH almost coincides with the vertical $C'c'$; practically it does coincide because the line is so short. The perpendicular laid off at the end of λ_3 is a line through c' perpendicular to the diagonal cH, and these two perpendiculars intersect at the point H' which practically coincides with c'. The rest of the construction is sufficiently indicated in the diagram. The right half of the displacement diagram belongs to the left half of the arch. In the same manner the left half of the displacement diagram is drawn, the last point to be located being b'.

Since the hinges a and b are fixed in position, a'' and b'' coincide with a' and b' respectively, and as the hinge c moves in two arcs whose centers are at a and b on account of the rotation of the segments about a and b, the lines $a''c''$ and $b''c''$ are drawn perpendicular to the radii ac and bc. Their intersection locates c'' which is found to be just a little on the right of the vertical through c'. If the truss were entirely symmetrical the resultant displacement of the hinge c would be vertical, and then only one-half of the displacement diagram would be required.

The diagram $a''A''C''c''$, similar to the left half of the truss diagram is next constructed. That for the other half is omitted in order to avoid confusion. The resultant displacement of any panel point as C is indicated by the line from C'' to C' both in magnitude and direction. The horizontal component of the displacement of A is 0.110 inch.

The deflections of the upper panel points, expressed in inches, are as follows:

	C	H	G	F	E	D
Left half of arch . .	1.183	0.825	0.572	0.388	0.241	0.120
Right half of arch .		0.837	0.589	0.405	0.257	0.130

Those of the lower panel points are:

	c	h	g	f	e	d
Left half of arch . .	1.175	0.815	0.558	0.366	0.210	0.087
Right half of arch .		0.827	0.574	0.382	0.224	0.094

The deflections of A and B are 0.020 and 0.021 inches.

Fig. 107 gives the displacement diagram due to a change in temperature. As the changes in length of all the members are directly proportional to their lengths for a given change in temperature, the portion $c'C'A'a'$ of the diagram is similar to the truss diagram. It was constructed by laying off $c'a'$ equal to the computed shortening of the chord ac (Fig. 104) for a fall in temperature of 75 degrees below the standard of 50 degrees Fahrenheit. The length of ac is 107.703 feet, and the coefficient of expansion of steel is 0.0000065, making $\lambda = 0.630$ inches. The hinge c deflects in a vertical line and therefore c'' is located at the intersection of the vertical $c'c''$ and the line $a''c''$ which is perpendicular to the radius ac. The measurement of the original diagram gave the deflection of c as 1.696, and that of C as 1.731 inches. The shortening of Aa is 0.262 and of Bb is 0.276 inches. The horizontal component of the displacement of A is 0.655 inches.

At the maximum temperature of 125° F. and with no live load on the bridge the center hinge c is 1.696 inches above its normal position at the standard temperature; while at the minimum temperature of $-25°$ F. and with a full live load on the bridge, the center hinge is $1.696 + 1.175 = 2.871$ inches below the normal position, thus giving an extreme range of deflection of 4.567 inches. The panel point C has a range of 4.645 inches, which exceeds the average of the ranges of A and B by 4.396 inches. The range of the horizontal movement of A is $2 \times 0.655 + 0.110 = 1.42$ inches.

Since the special equilibrium polygon for any given loading must always pass through the three hinges, the form of the polygon is affected by the rise or fall of the central hinge due to temperature changes, and this modifies the value of H, and consequently the stresses in all the members of the arch. As the change in elevation of the center hinge is relatively small for the usual ratios of the rise to span, the corresponding effect on the stresses is in most cases so slight that it is customary not to consider it.

Prob. 73. Find the deflection of the center hinge due to the live load, provided the rise of the arch in Fig. 104 is increased from 40 to 47.5 feet, the depth at the crown, the grade of the roadway and the cross-section of all its members remaining the same.

ART. 74. INFLUENCE LINES.

As it is intended to use influence lines to determine the loading for two-hinged arches and the resulting stresses, it seems desirable to present their application also to three-hinged arches in order that the student may be able to compare their use with the methods already given, and to receive the aid of the latter in interpreting the significance of the various forms of the influence diagrams, as well as to consider their relative advantages. It may be added that influence lines were not employed in Part II because it is believed that for simple trusses their use in the determination of the position of the live load is not as convenient as that of the methods given.

An influence line of a truss is one which shows the variation of a stress, moment, shear, reaction, or any other function of a truss, or of any of its members, when a given load moves across the structure. Influence lines are sometimes employed to find the position which any specified loading must occupy in order to produce the maximum or minimum value of the stress or other function. They may also be used to find this value. The load usually employed for this purpose is unity. The ordinate to the influence line at any point represents the value of the function when the load unity is at the position indicated by the point, and for any other load occupying the same position, the value of the ordinate is multiplied by the given load.

Let it be required to construct the influence line for the stress S in the diagonal of Fig. 108 which is cut by the given section. If a load P be placed on the left half of the arch the reactions of the supports will be R_1 and R_2 (Art. 60). Since there is no

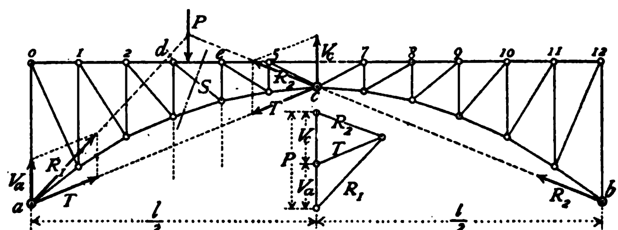

Fig. 108.

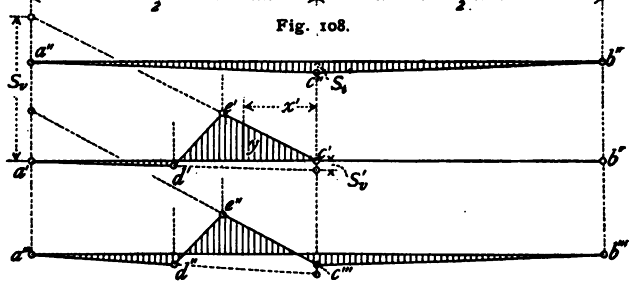

Fig. 109.

load on the right of the center hinge, the reaction at c against the left half of the arch is equal to R_2. Now let R_1 be resolved into the components V_a and T, the former being vertical and the latter directed toward the center hinge c. Similarly, R_2 is resolved into the vertical V_c and the component T which is directed toward the hinge a. The resolution is made in the force diagram, by drawing the ray T parallel to the line ac. Since the rays are respectively parallel to the three sides of the equilibrium triangle whose closing side is ac, the vertical components V_a and V_c are equal to the vertical reactions at a and c of the left half of the arch when regarded as a simple truss subject to the load P. The component T is equal to the reaction at a when a load equal in magnitude to V_c is placed on the arch at the center. The stress S is therefore equal to the sum of the stresses obtained under both of the preceding conditions, and the influence line for the stress S is equal to the combined influence lines for the same conditions.

It is first required then to construct the influence line for a load unity traversing the left half of the arch when acting as a simple truss with vertical reactions at a and c. Let S_v be the stress in the given diagonal when a load is placed at the panel point 5 that is of such a magnitude as to produce a vertical reaction at a equal to unity, and let S'_v be the stress when a load is placed at 1, so as to produce a vertical reaction at c equal to unity. When a load unity is placed at a distance x' from the right support c, the left reaction is $V_a = x'/\frac{1}{2}l$, $\frac{1}{2}l$ being the span. If the load is on the right of the panel cut by the section there is only the external force V_a on the left, and hence the stress in the diagonal is $V_a \times S_v = \frac{2x'}{l} S_v = y$. On the horizontal axis $a'c'$ in Fig. 109 let the ordinate S_v be erected at a' and its extremity joined with c', then the value of the ordinate y will satisfy the preceding equation and $e'c'$ will be the influence line on the right of the panel point e. Similarly $a'd'$ is found to be the influence line on the left of panel point d. S_v is laid off upward or positive as it is tension (+), while S'_v is compression (−) and is hence laid off downward.

In order to determine the form of the line $d'e'$, let the load P which rests on the stringer de be replaced by the panel loads P_3 and P_4. If the load is a distance g from the panel point e (=4) and the panel length is p, $P_3 = Pg/p$, and $P_4 = P(p - g)/p$.

Since the influence of the load P equals the sum of the influence of its components P_3 and P_4,

$$Py = P_3y_3 + P_4y_4,$$

in which y, y_3 and y_4 are the ordinates directly below P, P_3 and P_4 respectively. By substituting the values of P_3 and P_4 and making P equal to unity, there is obtained the equation,

$$y = \frac{g}{p}y_3 + \frac{p-g}{p}y_4,$$

which being of the first degree proves that the line $d'e'$ in Fig. 109 is a straight line.

Since the component V_a of the left reaction due to a load on the right half of the arch equals zero, the line of influence for that portion coincides with the axis $c'b'$. The complete line of influence is therefore $a'd'e'c'b'$.

The second line of influence required is that due to the component of the load unity whose magnitude equals V_c and which is applied at the center. The maximum ordinate is at c'' in Fig. 109, the load unity being then at the crown, and is equal to the stress in the diagonal for this loading. It is designated by S_t and its value is obtained by means of a stress diagram. Since V_c varies directly as the distance of the load from the nearest abutment the stress in the diagonal must vary in the same manner and hence the influence line for the entire span l is the line $a''c''b''$. This may also be seen by observing that the horizontal component of T in Fig. 108 is H, which according to Art. 60 varies as the ordinates of a triangle $a''c''b''$.

In order that the two influence lines may be properly combined let the points a', c' and b' be made to coincide with a'', c'' and b'' respectively, thus giving the lowest diagram in Fig. 109 as the final form.

For example, let the influence lines be drawn for the chord member U_4 and the diagonals D_4 and D_5 of the arch whose dimensions and loading were given in Art. 64. The stress diagram for a vertical reaction of unity at a is given in Fig. 94, Art. 66; that for a vertical reaction of unity at c is given in Fig. 111, while Fig. 112 is that for a load unity applied at the

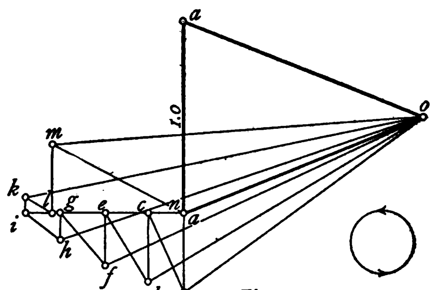

Fig. 112.

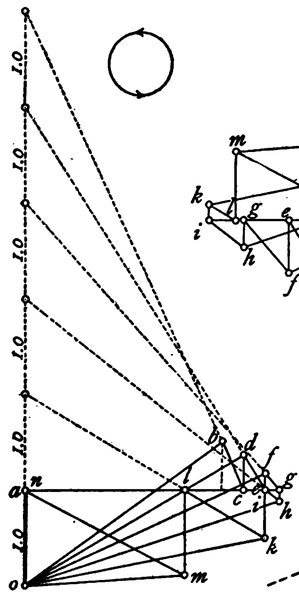

Fig. 111.

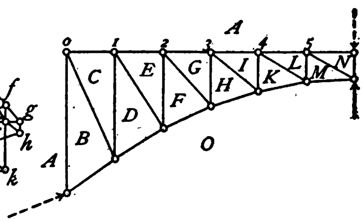

Fig. 110.

center. These diagrams give the following stresses for the above members:

	S_v.	S'_v.	S_t.
U_4	-5.00	-2.50	$+0.83$
D_4	$+3.02$	-0.19	-0.22
D_5	$+3.88$	-0.98	$+0.16$

These stresses are laid off as ordinates as shown in Fig. 113. It will be noted that they are laid off as positive and negative in accordance with their signs of tension and compression, and that the ordinate S'_v is laid off so as to be added algebraically to S_t. The construction may be tested by observing the following checks: The lines which are drawn to the extremities of the ordinates S_v and S'_v must intersect each other in the vertical drawn through the center of moments of the given member. The points where the line joining the extremities of the ordinates S_v and S_t crosses the axis must lie in the same vertical as

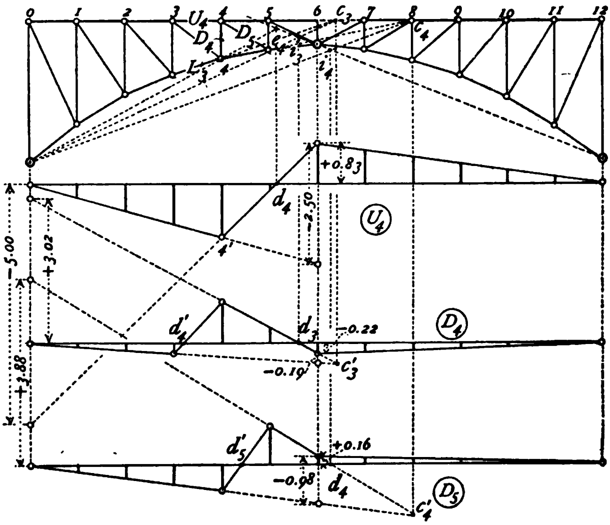

Fig. 113.

the points of division i and e found in Art. 63. Compare Fig. 113 with Figs. 89 and 90. The zero points d'_4 and d'_5 in Fig. 113 lie in the same verticals as the points of division in the fourth and fifth panels found by means of equilibrium polygons (triangles) as shown in Fig. 100, Art. 68.

The influence line for U_4 shows that if any concentrated load occupies a position to the left of d_4 it causes compression, while if it is on the right of d_4 it causes tension in U_4. To obtain the greatest compression in U_4 hence requires the panel points 0 to 5 inclusive to be loaded, while the greatest tension is due to the panel loads 6 to 12 inclusive. If it be desired to use the influence lines to determine the magnitude of the greatest compression in U_4 it may be done by measuring the ordinates under the panel points 0 to 5, and multiplying the sum, which is found to be -2.92, by the live panel load of 24 kips, giving a stress of -70.1 kips. If the diagram is carefully drawn the sum of the positive ordinates indicated by heavy lines will be equal to the sum of the negative ordinates, thus showing that under full load there is no stress in U_4. The summation of ordinates is most readily made by means of a pair of dividers.

The influence line for D_4 shows that the greatest tension is produced by panel loads 4 and 5 and the greatest compression by the remaining panel loads. The influence line for D_5 indicates the fact that there is only one point of division for the loads and that all the panel loads on the right of d'_5 produce the same kind of stress in D_5. It shows also that d_4, in the lowest diagram of Fig. 113, is not a real point of division when the corresponding point i_4 on the truss diagram is on the right of the center hinge.

If a uniform load be employed the greatest tension in U_4 may be obtained by multiplying the positive area of the influence diagram by the load per linear unit. If one foot be taken as the linear unit the area will be $\frac{1}{2} \times 102.9 \times 0.83 = 45.19$. The uniform load corresponding to the above panel load of 24 kips is $24/15 = 1.6$ kips per linear foot. Hence the stress is $45.19 \times 1.6 = 72.3$ kips. Panel loads are used in preference to a uniform load.

If a single excess panel load is specified, it must be placed at the position indicated by the largest ordinate for tension and compression respectively. If two equal excess panel loads are employed they should be so placed that the sum of the corresponding ordinates of the influence line shall be a maximum. For instance, if the excess loads are two panel lengths apart, they should be placed at panel points 6 and 8 (Fig. 113) for the greatest tension in U_4, and at 4 and 2 for the greatest compression. For the greatest tension in D_4 the excess loads should be placed at 4 and 6, while for the greatest compression the loads should be put at 6 and 8.

When locomotive wheel loads are specified their position for the greatest tension in U_4 is found by placing one of the wheels at the apex of the triangle as shown in Fig. 114 so that the pilot shall be near the point of division d_4. The stress due

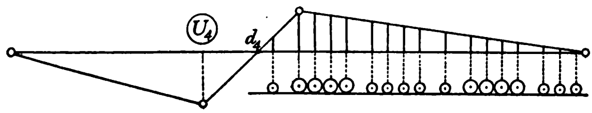

Fig. 114.

to each wheel load equals the product of the load and the ordinate directly above it, the total stress being the sum of these products. It is a question whether a greater stress may be obtained by placing the third wheel at the apex (which indicates the position of the center hinge), although this will place the pilot wheel a little to the left of d_4, and the only way to decide the question is to find the stress due to this position in the same manner as before and compare results.

It was proved in this article that the influence line is a straight line between the verticals which indicate the position of the floor beams. In case therefore that the bracing is of such a form that the center of moments for U_4 and the center hinge occupy the positions given in the upper diagram of Fig. 115, the influence line for U_4 must be modified as shown in the lower diagram.

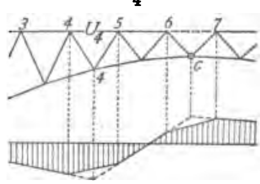

Fig. 115.

Prob. 74. Construct the influence lines for all the members of the truss in Fig. 91, and notice especially the diagrams for L_0, L_5 and D_2. Also compare those for D_6 and U_5.

CHAPTER VI.

TWO-HINGED ARCHES.

ART. 75. DESCRIPTIVE NOTES.

It was shown in Art. 73 that changes in temperature cause the larger part of the deflection of the three-hinged arch. By making the arch continuous at the crown the deflection is reduced, since the two halves of the arch are no longer free to turn at that point. The deformation of the arch due to changes in temperature is therefore partially prevented and this causes stresses in its members. Because the two-hinged arch is a stiffer structure it is better adapted for railroad traffic than the three-hinged arch, especially for long spans. In one of the common forms of two-hinged arched bridges the floor is supported by vertical posts, united by sway bracing, and resting upon arch ribs several of which are placed at comparatively short distances apart and thoroughly braced together by lateral and sway bracing. The posts are spaced so as to make the panels of the floor system equal and no diagonal bracing is placed between those resting on the same rib, but if they need longitudinal support on account of their length, lines of horizontal struts are extended from post to post until one end reaches the arch rib where it is securely fastened. An arch rib is a metallic girder built in the curved form of an arch and generally having the radial depth uniform throughout. The flanges are usually formed of angles and plates similar to the plate girder construction. The flanges are connected either by a solid web or by diagonal bracing.

The largest arch ribs with solid webs are those of the Washington bridge over the Harlem river at New York City, completed in 1889. The clear span is 510 feet and the clear rise 91.8 feet, while the span center to center of end hinges is 508.8 feet and the rise to the axis of the rib 89 feet 9.88 inches. The ribs are 13 feet deep and the web plate has radial stiffeners. The pins at the skewbacks are 18 inches in diameter. The floor system has 34 equal panels. A brief historical sketch of the bridge with illustrations of the competitive designs submitted, as well as that finally adopted, may be found in Engineering News, Dec. 27, 1890. A complete description of its construction, containing numerous views and detail drawings, is published in a volume entitled 'The Washington Bridge,' by WILLIAM R. HUTTON, the chief engineer.

The two-hinged ribs of the Minneapolis steel arched bridge have a span of 258 feet, a rise of 26 feet, and a depth of about 5 feet, while those at Lansing, Michigan, have a span of 110 feet, a rise of 13 feet, and a depth of 4 feet. A short illustrated description of the latter bridge is given in Engineering News, Nov. 14, 1895.

In the steel arch, which replaced the Niagara Falls and Clifton suspension bridge in 1898, the chords of the ribs are connected by diagonal bracing. This structure not only has the largest span of any trussed arch with parallel chords but that of any kind of arch in the world. The span is 840 feet center to center of end hinges, and the rise from the level of the hinges to the center of the trusses at the crown is 150 feet. The trusses are 26 feet deep. See Railroad Gazette, Feb. 28, 1896.

The Riverside Cemetery bridge at Cleveland, Ohio, erected in 1896 has an arch span of 142 feet. The rise of the parabolic arch ribs is 27 feet, their depth being 2 feet at the ends and 5 feet at the crown. The roadway is carried by each rib at only three points: at the crown and the haunches. The middle half has a lattice web, while the ends have solid web plates.

The Garabit viaduct in France contains a crescent-shaped trussed arch with two hinges, the span being 165 meters, the mean rise 65 meters, and the depth at the crown 10 meters. See Engineering News, Aug. 9 and 30, 1884. The bridge at Gruenenthal over the North Baltic ship canal has the same shape, the middle portion of the floor being suspended while the ends are supported by posts resting on the arches. The span is 513 feet, mean rise 77 feet, and crown depth 13.4 feet.

Another principal type of the two-hinged arch, and perhaps the most prevalent one, has spandrel bracing. The best example of this type is the steel arch which replaced the railroad suspension bridge near Niagara Falls in 1897. Its span is 550 feet, the rise of the lower chord 113.3 feet, and the depth at the crown 20 feet. The end shoe is so arranged as to form practically a pin with a diameter of about nine feet, and with roller friction. A short description with general plans of the structure, together with historical notes on the bridges which formerly occupied this site was published in the Engineering News, Aug. 6, 1896. See also the same periodical for April 22, 1897, as well as the illustrated descriptions in the Railroad Gazette, April 24, 1896, and the Engineering Record, April 24, 1897. Both of the steel arches over the Niagara river were designed by LEFFERT L. BUCK.

Few large arches for roofs have been erected which are without the hinge at the crown. Those of the Grand Central station of the N. Y. C. & H. R. R. R. in New York belong to this class. The span is 199 feet 2 inches, and the rise 94 feet.

Prob. 75. With the aid of indexes to the engineering periodicals, make as complete a list as possible of metallic arches with two hinges, classifying the different types, and recording the span, rise, and ratio of rise to span in each case.

ART. 76. REACTIONS FOR AN ARCH RIB.

The stresses in any arch rib are determined from the bending moments and shears and these are to be computed by means of the reactions of the supports. Arts. 76–85 will be devoted to the discussion of arch ribs with two hinges.

Let l be the span of a two-hinged arch rib and h the rise of its crown. Let a load P be situated at any distance kl from the left support, k being any fraction less than unity. This load is held in equilibrium by the two inclined reactions R_1 and R_2 whose lines of action must intersect that of P at a common

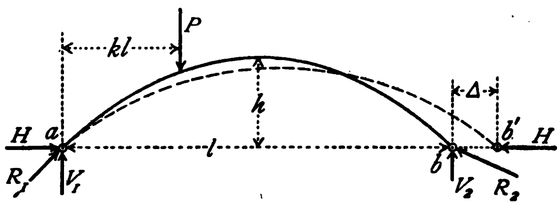

Fig. 116.

point. The reaction R_1 may be replaced by its vertical component V_1 and horizontal component H, and likewise R_2 is given by its components R_2 and H. Here H is the horizontal thrust at the hinges due to P; it is the same at both hinges because the sum of the horizontal forces acting on the structure must equal zero.

The vertical forces V_1 and V_2 are found by taking moments successively about the supports a and b; thus

$$V_1 = P(1 - k), \qquad V_2 = Pk, \qquad (1)$$

or, the vertical components of the reactions are the same as the reactions for a simple beam.

For the three-hinged arch the horizontal thrust H was determined by the condition that the line of action of R_2 must pass through the hinge at the crown (Art. 60). For the two-hinged arch, however, the value of H cannot be found by pure statics, but a condition must be introduced based upon the elastic properties of the material. The two-hinged arch is indeed similar to a continuous beam of two spans in regard to the determination of reactions; in both cases three unknown reacting forces are to be found, while the principles of statics furnish but two conditions.

Let the arch rib in Fig. 116 be supposed to be placed on rollers at the end b, so that when P causes a deflection of the rib the end b moves horizontally to b'. In this condition of things there is no thrust H. Let Δ be the horizontal displacement bb' thus produced. Now suppose a horizontal force H to be applied at b' which is sufficiently large to bring b' back to b. Then the value of Δ due to P is equal to the value of Δ produced by H. This is the condition by which the horizontal thrust H is determined. It is now proposed to find expressions for these two values of the displacement Δ.

The deformation of an arch rib, like that of a beam, is due mainly to flexure. The flexural stresses, when the elastic limit is not exceeded, are proportional to their distances from a neutral surface upon which there is no stress due to flexure. To find the horizontal displacement Δ due to P, let a horizontal force unity be applied at b in the direction of bb'. The external work overcome in the displacement is then $\frac{1}{2}(1 \times \Delta)$ and this is equal to the internal work of the flexural stresses. Let ds in Fig. 117 be an elementary length of the arch rib and I the moment of inertia of its cross-section about the neutral axis. Let M' be the bending moment of the vertical forces, which produces a unit-stress $M'c/I$ on the remotest fiber, c being the distance of that fiber from the neu-

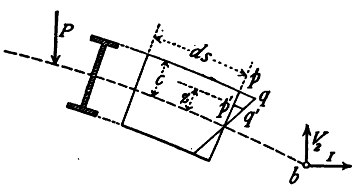

Fig. 117.

tral axis. The elongation or shortening of this fiber is $\frac{M'c}{I}\cdot\frac{ds}{E}$ which is represented by pq, and that of any other fiber distant z from the neutral axis is $\frac{M'z}{I}\cdot\frac{ds}{E}$. Now let m be the moment due to the horizontal force unity at b. The unit-stress due to this on the fiber $p'q'$ is mz/I and if a be the area of that fiber the total stress on it is maz/I. The internal work of this fiber is accordingly $\frac{M'maz^2ds}{2EI^2}$, and the summation of this over the entire cross-section is effected by putting $\Sigma az^2 = I$. The total internal work done by the load P in the entire arch rib is then $\int\frac{M'mds}{2EI}$ if the integral be extended over the entire span. Equating this to the external work, gives

$$\Delta = \int\frac{M'm\cdot ds}{EI} \qquad (2)$$

for the horizontal displacement of b due to the load P.

In a similar manner let M'' be the moment due to the horizontal thrust H, or $M'' = -Hm$. Then by similar reasoning (see Mechanics of Materials, Art. 109),

$$\Delta = \frac{1}{H}\int\frac{M''^2ds}{EI} = H\int\frac{m^2ds}{EI} \qquad (3)$$

is the horizontal displacement of b due to the thrust H.

Equating the two values of Δ expressed by (2) and (3) gives the condition that the hinge b cannot move horizontally under the action of the load P. Hence,

$$H = \int\frac{M'mds}{EI} \Big/ \int\frac{m^2ds}{EI} \qquad (4)$$

is the formula for determining the thrust of a two-hinged arch under the action of flexural stresses.

Prob. 76. A cantilever beam of length l has a load P at the free end and the equation of the elastic curve is

$$6EIy = P(3l^2x - x^3)$$

when referred to an origin at the free end. Show from the preceding formula (2) that the horizontal displacement of the free end is $2P^2l^5/15E^2I^2$.

ART. 77. PARABOLIC ARCH RIB.

Let the curve in Fig. 118 represent a solid arch rib of parabolic form, the vertex of the parabola being at the crown. The equation of the parabola, referred to the hinge a as an origin, is

$$y = 4h\left(\frac{x}{l} - \frac{x^2}{l^2}\right), \qquad (1)$$

and from this the ordinates y may be computed for all values of x. A single load P is placed on this arch at a distance kl from the left end. By the last article the vertical reactions due

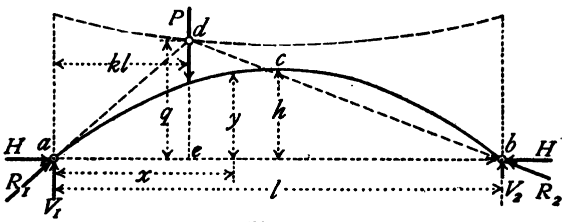

Fig. 118.

to P are $V_1 = P(1 - k)$ and $V_2 = Pk$. It is required to find the value of the horizontal thrust H.

The deformation under the action of the exterior forces is due mainly to flexural stresses. At any section on the left of P the bending moment is $M = V_1x - Hy$, and at any section on the right of P it is $M = V_1x - P(x - kl) - Hy$. Let M' represent the moment due to the vertical forces and M'' that due to the horizontal thrust H. Then M' has the value V_1x or $V_1x - P(x - kl)$ and M'' has the value $-Hy$. In general, y is the moment due to a horizontal force unity acting away from b, and

$$M = M' + M'' = M' - Hy$$

gives the bending moment due to all the external forces.

Formula (4) of the last article may now be applied to the determination of the thrust H. In order to simplify the work very materially let it be assumed that the moment of inertia I of the rib cross-section varies from the crown to the skewback hinges as the secant of the angle of inclination of the axis of the rib. If I_c be the moment of inertia at the crown then $I = I_c \sec i$. Moreover $ds = dx \cdot \sec i$. Substituting the values of M, m, I and ds, the formula for H becomes

$$H = \frac{\int_0^{kl} P(1-k)xydx + \int_{kl}^{l} (P(1-k)x - P(x-kl))ydx}{\int_0^l y^2 dx}$$

and, inserting the value of y from (1) and performing the integrations, this reduces to

$$H = \frac{5}{8}\frac{Pl}{h}(k - 2k^3 + k^4) \qquad (2)$$

which is the thrust of the parabolic two-hinged arch due to the flexural stresses produced by a single load P.

Referring again to Fig. 118 let d be the point where the lines of action of R_1, P and R_2 intersect. As P moves across the span d generates a curve called the "reaction locus." The abscissa of d is kl and its ordinate will be called q. To determine q it is only necessary to note that in the triangle ade the sides ae and de are proportional to H and V_1, or

$$q = \frac{V_1 kl}{H} = \frac{1.6h}{1 + k - k^2}. \qquad (3)$$

By plotting the reaction locus from this equation the directions of R_1 and R_2 may be found graphically for a load in any position by connecting the hinges with the point where the vertical through the load intersects the curve. After the directions of the reactions are known their magnitudes are readily determined by means of the force triangle. The locus of d is called the reaction locus because of its use in determining reactions.

For a uniform load w per horizontal linear unit the thrust H may be obtained by substituting for P the value $w \cdot d(kl)$ in (2) and integrating with respect to k between the limits 0 and 1. This gives

$$H = \frac{5\,wl^2}{8\,h}\int_0^1 (k - 2\,k^3 + k^4)\,dk = \frac{wl^2}{8\,h} \qquad (4)$$

which is the same as that found in Art. 43 for the suspension cable and in Art. 65 for the three-hinged arch. In this case the equilibrium polygon coincides with the arch axis and hence there is no bending moment in any section of the rib due to a uniform load over the entire span. The student should observe that this determination of H implies that the load is not merely uniformly distributed on the floor of the bridge but is transferred to the arch rib as a uniform load per horizontal linear unit. With the usual construction the load is transferred to the rib as a series of horizontally equidistant panel loads.

Let a parabolic arch rib be taken whose span is 258 feet and whose rise is 26 feet. It is represented in Fig. 119, and the weight of the horizontal roadway and its loads is transmitted by equidistant vertical columns to the rib which is thereby divided

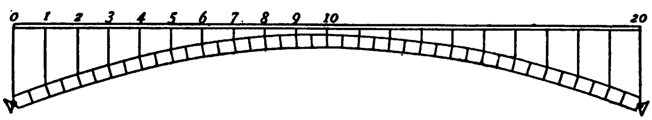

Fig. 119.

into twenty parts. The dead and live panel loads are 59.0 and 18.2 kips respectively. With the aid of a table of squares and cubes let the values of V_1, V_2 and H be computed for a panel load of 1 kip, and the results placed in the following table. The values of H for the loads 10 to 19 are the same as those for 10 to 1 inclusive.

Load at	V_1	V_2	H
1	0.95	0.05	0.3089
2	0.90	0.10	0.6084
3	0.85	0.15	0.8918
4	0.80	0.20	1.1511
5	0.75	0.25	1.3812
6	0.70	0.30	1.5759
7	0.65	0.35	1.7322
8	0.60	0.40	1.8457
9	0.55	0.45	1.9152
10	0 50	0.50	1.9381
1 – 19	9.50	9.50	24.759

For the live panel loads 1 to 8 inclusive, $V_1 = 6.20 \times 18.2 = 112.84$, $V_2 = 1.80 \times 18.2 = 32.76$, and $H = 9.4952 \times 18.2 = 172.81$ kips. The value of H for the dead load is $24.759 \times 59 = 1460.78$ kips.

If the load were uniformly distributed over the span the value of H would be given by equation (4) in this article, as $(59 \times 20 \times 258)/(8 \times 26) = 1463.66$ kips. If this were the value of H for the dead panel loads the vertices of the special equilibrium polygon would lie on the parabolic axis of the arch at the sections under the equidistant loads, but as H is 2.88 kips less than this there will be a positive bending moment in the rib at any section equal to $2.88y$ kip-feet, y being the corresponding ordinate to the parabolic axis expressed in feet. The values of y (see Fig. 120) and of the bending moment M are as follows:

Section	0	1	2	3	4	5
y	= 0	4.94	9.36	13.26	16.64	19.50
M	= 0	+ 14.2	+ 27.0	+ 38.2	+ 47.9	+ 56.2

Section	6	7	8	9	10
y =	21.84	23.66	24.96	25.74	26.0 feet
M =	+ 62.9	+ 68.1	+ 71.9	+ 74.1	+ 74.9 kip-feet

Prob. 77. An arch rib, like Fig. 119, has a span of 125 feet and a rise of 25 feet. The floor system has 10 equal panels and the dead and live panel loads are 50 and 15 kips respectively. Prepare a table of reactions like the one given in this article and compute the bending moments for the dead load.

ART. 78. POSITION AND MOMENTS FOR LIVE LOAD.

In Fig. 120 the axis of the arch whose dimensions were given in Art. 77 is drawn, the ordinates y being however laid off to twice the scale used for the horizontal distances. The curved

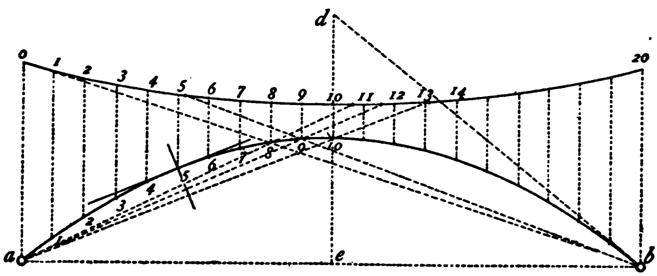

Fig. 120.

reaction locus is also given, the ordinates q being computed by formula (3) in Art. 77. The values of q for the points 0 to 10 inclusive are 41.60, 39.71, 38.17, 36.90, 35.86, 35.03, 34.38, 33.89, 33.55, 33.35, and 33.28 feet. The curve is symmetrical with respect to a vertical at the center and its ordinates are laid off with the same scale as that used for y.

The reaction locus is used in exactly the same way as the rectilinear reaction locus in Figs. 89 and 90, Art. 63, for the three-hinged arch. For example, the greatest positive moment in the arch at 9 is produced by the live panel loads 6–11 inclusive, and the greatest negative moment by loads 1–5 and 12–19. The loads at 0 and 20 cause no stresses in the arch. As the line through b and 8 almost touches the reaction line at 0, the greatest negative moment for each of the sections 1 to 8 inclusive is due to the loads on the right of the line through the left hinge a and the corresponding center of moments, while the loads on the left cause the greatest positive moment.

The live load moments are most conveniently found by computation. Thus for section 5, the greatest positive moment is due to loads 1–8, and equals

$$M_5 = 112.84 \times 64.5 - 172.81 \times 19.50$$
$$- 18.2(1 + 2 + 3 + 4)\,12.9 = + 1560.6 \text{ kip-feet.}$$

For the section at the crown the loading includes 7–13 and the greatest positive moment is

$$M_{10} = 63.70 \times 129 - 12.9243 \times 18.2 \times 26.0$$
$$- 18.2(1 + 2 + 3)\,12.9 = + 693.8 \text{ kip-feet.}$$

If both the positive and negative moments are computed the results may be checked by finding the moments for a full live load as explained in Art. 77 for the dead load.

Prob. 78. Find the position of the live load which causes the maximum moments at sections 1 to 5 of the arch in Prob. 77 and compute the moments at sections 2 and 5 due to the live panel load of 15 kips.

ART. 79. THE AXIAL THRUST.

It is next required to find the thrust in the direction of the axis at each section and this may best be done by finding the thrust at each section due to a load of 1 kip at each panel point successively and tabulating the results. In Fig. 121 the computed values (Art. 77) of V_1, V_2 and H are laid off to scale for a load of 1 kip at points 1–10, and 15 of the arch in Fig. 120. For instance, for the load at 5, *ea*, *be* and *e5* represent V_1, V_2 and H respectively; while *b5* and *5a* give the magnitudes and directions of R_2 and R_1. These directions are the same as those obtained by joining the hinges *b* and *a* with the point 5 on the reaction locus in Fig. 120 provided the vertical scale were equal to the horizontal scale in that diagram. As the reactions are to be projected on a number of tangents in order to obtain the thrusts it is important that the points 1, 2, 3, etc., in Fig. 121 be located accurately. Let *ad* be drawn parallel to the tangent to the parabolic axis of the arch at section 5. If the load be placed at 5 the thrust at section 5 is the projection of the left reaction $5a (= R_1)$ on *ad*, which equals $5'a$. By

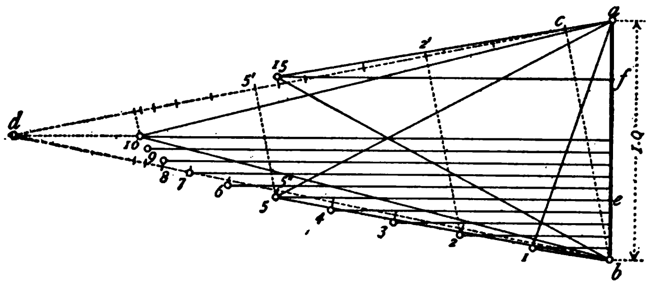

Fig. 121.

applying the scale of force it is found to be 1.501 kips. The thrusts due to the loads 6 to 10 inclusive are found in the same manner. For the load at 15 the values of $V_1 = fa$, $V_2 = bf$ and $H = 15 - f$ are respectively equal to $V_2 = be$, $V_1 = ea$, and $H = 5e$ for the load at 5. It is evident therefore that the projection of the left reaction $15 - a$ on *ad* is equal to the projection $b5''$ of the right reaction $b5$ on *bd*. Its value is 1.404 kips. It will materially simplify the diagram to use the points 1 to 9 instead of 11 to 19 for the loads on the right of the crown and to avoid errors it is desirable to add the numbers 11 to 19 in parentheses directly below 9 to 1 in Fig. 121.

For a load on the left of 5, as, for example, that at 2, the thrust at section 5 equals the projection $2'a$ of the left reaction $2a$ minus the projection *ca* of the load, which is shown by the diagram to be equal to $2'c$ the projection of the right reaction $b2$ on *ad*. Its value is 0.576 kip.

In this manner the thrusts at all the sections in the left half of the arch are found and placed in the following table. The long dashes in the table indicate the position of the section with respect to the loads, and separates the thrusts to be measured respectively from *c* and from *a* in the diagram. The short dashes indicate the points of division for the live load. For sections 0 to 8 the greatest positive moment is due to the loads above the corresponding dashes, and for sections 9 and 10 it is due to the loads between the upper and lower dashes. In order to obtain the direction of the tangents accurately it is well to draw a series of lines radiating from the left hinge which shall intercept distances on the ordinate at mid-span equal to the quotient obtained by dividing double the rise by the number of panels in the half-span. The tangent at 0 cuts off an ordinate at mid-span equal to double the rise.

AXIAL THRUST DUE TO A PANEL LOAD OF 1 KIP.

LOAD AT	THRUST AT SECTION.										
	0	1	2	3	4	5	6	7	8	9	10
1	0.642	0.614	0.278	0.283	0.287	0.293	0.295	0.300	0.300	0.305	0.309
2	0.900	0.878	0.853	0.557	0.567	0.576	0.582	0.590	0.595	0.602	0.608
3	1.146	1.129	1.110	1.091	0.833	0.846	0.856	0.867	0.876	0.885	0.892
4	1.367	1.356	1.341	1.326	1.308	1.089	1.104	1.119	1.130	1.142	1.151
5	1.561	1.554	1.545	1.533	1.518	1.501	1.323	1.341	1.354	1.369	1.381
6	1.723	1.720	1.715	1.706	1.696	1.683	1.668	1.529	1.545	1.562	1.576
7	1.849	1.851	1.849	1.844	1.837	1.826	1.815	1.798	1.698	1.717	1.732
8	1.936	1.940	1.942	1.940	1.936	1.928	1.919	1.904	1.888	1.828	1.846
9	1.981	1.987	1.992	1.993	1.991	1.985	1.978	1.967	1.953	1.935	1.915
10	1.986	1.994	1.999	2.002	2.003	2.000	1.995	1.985	1.974	1.957	1.938
11	1.945	1.955	1.963	1.967	1.968	1.967	1.963	1.956	1.946	1.932	1.915
12	1.862	1.874	1.881	1.887	1.889	1.890	1.887	1.882	1.874	1.860	1.846
13	1.738	1.750	1.759	1.764	1.767	1.769	1.766	1.763	1.756	1.745	1.732
14	1.575	1.586	1.594	1.600	1.604	1.605	1.604	1.602	1.595	1.586	1.576
15	1.375	1.386	1.393	1.399	1.403	1.404	1.405	1.403	1.398	1.390	1.381
16	1.144	1.153	1.159	1.164	1.168	1.170	1.170	1.168	1.165	1.159	1.151
17	0.885	0.892	0.898	0.902	0.905	0.907	0.907	0.905	0.903	0.898	0.892
18	0.602	0.607	0.610	0.613	0.615	0.616	0.616	0.615	0.614	0.611	0.608
19	0.306	0.308	0.310	0.312	0.312	0.313	0.313	0.312	0.312	0.310	0.309
Total	26.523	26.534	26.191	25.883	25.607	25.368	25.166	25.006	24.876	24.793	24.759

To obtain the thrust in each section due to the dead load of 59 kips the quantities in the line marked 'total' must be multiplied by 59. As the greatest positive moment at section 5 is due to the live panel loads of 18.2 kips each at points 1 to 8 inclusive, the corresponding thrust is readily found from the table to be $9.742 \times 18.2 = 177.3$ kips. For the section at the crown the required thrust due to the live load is

$$12.924 \times 18.2 = 235.2 \text{ kips.}$$

Prob. 79. Prepare a table similar to the above for the arch rib in Prob. 77.

ART. 80. RIB SHORTENING.

The direct effect of the thrust along the axis is to shorten the axis of the rib. It would also shorten the span provided one end were free to move but as this is not the case it will develope equal and opposite negative reactions H. The horizontal displacement due to H must be equal to that due to the shortening of the rib (Art. 76).

The shortening λ of a differential portion of the axis ds which is under a compressive unit stress s equals sds/E, the horizontal component of which is sdx/E, and hence the shortening of the span under the conditions named above is

$$\Delta = \int_0^l \frac{sdx}{E}.$$

To find the true value of this integral would require the variable compressive unit-stress s to be expressed in terms of x, but as its variation is yet unknown, an approximate result may be obtained by using an average value of s which can be considered as constant. The shortening of the span then becomes $\Delta = sl/E$. By the formula (3) of Art. 76, and making the same assumptions regarding ds and I as in Art. 77,

$$H = \frac{-\Delta}{\int \frac{m^2 ds}{EI}} = -\frac{sl}{\int \frac{m^2 dx}{I_c}}. \qquad (1)$$

Here m denotes the moment due to a horizontal force of unity acting outward at the hinge b (Fig. 116) or $m = y$. Inserting for y its value for the parabola from (1) of Art. 76, and integrating between the limits 0 and l, there results,

$$H = -\frac{15\, sI_c}{8\, h^2} \tag{2}$$

which is the horizontal thrust due to the direct compression along the axis of the arch rib. The minus sign here denotes that the H is directed outward, away from the hinge, or that it prevents the reduction of the span which the rib shortening tends to produce.

The above formula contains the two quantities s and I_c which are unknown in making a design, for their values must be such as are required to include the thrust due to changes in temperature. It is necessary therefore to obtain these values by trial. Let it be assumed that as the result of several approximations for the arch rib used as an example in the preceding articles, the average area of the combined cross-section of both flanges was found to be 185 square inches, and the moment of inertia I_c at the crown about 170 100 inches^4. In determining the average area of the flanges it was assumed that in the construction the area would only be changed at sections 2, 4, 6, 8, etc. (Fig. 119).

In order to find the approximate value of s in equation (2) which is due to the axial thrust alone it is next required to find the average thrust due to the loading which makes the stress in either flange a maximum. By rewriting the data in the table in the preceding article a table may be prepared whose headings are given below in this paragraph, while the first column at the left contains the numbers 0 to 20 of the sections. Instead of inserting only half the values in the lines for the end sections 0 and 20 the full values may be inserted in only one of them. The approximate average thrust throughout the rib produced by each load may be found by adding the columns in this table and dividing the sum by 20. The results are as follows:

PANEL LOAD OF 1 KIP AT

Section	1	2	3	4	5	6	7	8	9	10
or section	19	18	17	16	15	14	13	12	11	
Average thrust,	0.327	0.634	0.926	1.190	1.428	1.621	1.780	1.895	1.954	1.987

The average thrust due to the dead panel load of 59 kips is therefore $25.497 \times 59.0 = 1504.3$ kips. As the live panel loads 1 to 8 produce the greatest positive bending moment at section 5 the corresponding average thrust is $9.801 \times 18.2 = 178.4$ kips. For the section at the crown the panel loads 7 to 13 cause an average thrust of $13.245 \times 18.2 = 241.1$ kips. That due to the specified maximum change in temperature (to be determined in the next article) causing the greatest positive moment is -40.4 kips.

Let it now be required to find the value of H due to the shortening of the rib under the combined average thrust of the loading which makes the flange stress a maximum at the crown. This thrust is $1504.3 + 241.1 - 40.4 = 1705.0$ kips, which gives a stress of $1705/185 = 9.22$ kips per square inch. By equation (2), H is found to be -30.2 kips, and the bending moment $30.2 \times 26.0 = +785.2$ kip-feet. The thrust equals H at the crown, or -30.2 kips. The average thrust due to this H of -30.2 kips again modifies the value previously obtained but as it is relatively small it is frequently not necessary to re-compute the value of H. As for any given arch quite a number of values are to be found for H due to thrust, it will facilitate the computation by first calculating its value for $s = 1.0$ kip and $I_c = 1$ inch^4.

It will be observed that the larger part of the rib shortening is due to the dead load. This might be eliminated if, during erection, the span could be shortened after the ribs are properly connected at the center so that when the full dead load is in place the span and rise will equal, at the standard temperature, the values assumed in computing the stresses.

Prob. 80. Compute the bending moment and thrust due to rib shortening at section 5 in the preceding example.

ART. 81. INFLUENCE OF TEMPERATURE.

Changes in temperature change the value of H but do not affect V_1 or V_2. It is usually specified that an arch shall be designed to be subject to a variation of ± 75 degrees from the standard temperature of 50 degrees Fahrenheit. Let ϵ be the coefficient of expansion and t the rise in temperature, then the span l will be increased by ϵtl provided one end is free to move. As both hinges are fixed in position when the supports do not yield, equal and opposite positive reactions H are produced and consequently negative bending moments throughout the arch. The value of H must be such as to prevent the horizontal displacement ϵtl, which corresponds to Δ in Fig. 116. From the equation (3) of Art. 76,

$$\Delta = H\int\frac{m^2ds}{EI}$$

the value of H is found by making $\Delta = \epsilon tl$, $m = y$, $ds = dx \sec i$, $I = I_c \sec i$ and integrating between the limits 0 and l. Thus

$$EI_c\epsilon tl = H\int_0^l y^2dx = \frac{16\,Hh^2}{l^4}\int_0^l (lx - x^2)^2dx$$

and by integrating and solving for H, there results for a rise in temperature,

$$H = +\frac{15\,EI_c\epsilon t}{8\,h} \qquad (1)$$

and similarly for a fall in temperature,

$$H = -\frac{15\,EI_c\epsilon t}{8\,h} \qquad (2)$$

For the steel arch under consideration, with $E = 26\,000\,000$

pounds or 26 000 kips per square inch, $I_c = 170\ 100$ inches^4, $\epsilon = 0.0000065$, $t = \pm 75°$ and $h = 26$ feet or 312 inches, H is found to be ± 41.53 kips. The positive moments occur under falling temperature and are obtained by multiplying H by the values of y given in Art. 77 for the different sections. The results for sections 0 to 10 (Fig. 120) are as follows: 0, 205.1, 388.7, 550.5, 690.8, 809.6, 907.0, 982.6, 1036.5, 1069.0, and 1079.7 kip-feet.

By means of a diagram similar to Fig. 121 the thrust at the sections 0 to 10 due to $H = 1$ kip are found to be: 0.926, 0.939, 0.951, 0.962, 0.972, 0.980, 0.987, 0.992, 0.996, 0.999, and 1.000 kip, and the average thrust is 0.974 kip. For $H = 41.53$ kips the thrusts corresponding to the positive bending moments are -38.5, -39.0, -39.5, -39.9, -40.4, -40.7, -41.0, -41.2, -41.4, -41.5, and -41.5 kips, while the average thrust is -40.4 kips.

Prob. 81. Construct a diagram showing the forces acting upon the arch in the above example when the fall in temperature is a maximum. Draw the force diagram and the special equilibrium polygon, and indicate the closing line of the polygon.

ART. 82. FLANGE STRESSES.

The moments and axial thrusts at the crown of the parabolic arch rib under the influence of the dead load, live panel loads 7 to 13 (Fig. 120), a decrease in temperature of 75 degrees below the standard, and the shortening of the rib due to the thrust, and which together produce the maximum positive moment, were found in Arts. 77–81 to be as follows:

SECTION 10.

	BENDING MOMENT.	AXIAL THRUST.
Dead load	+ 74.9 kip-feet	1460.7 kips
Live load (7 – 13)	+ 693.8	235.2
Temperature	+ 1079.7	– 41.5
Rib shortening	+ 785.2	– 30.2
Total	+ 2633.6	1624.2

In computing the moment of inertia I_c an effective depth of 5 feet was assumed, and if the flanges are to take all the moment and the web all the shear, as usually specified, the maximum compression in the upper flange is

$$2633.6/5 + 1624.2/2 = 1338.8 \text{ kips},$$

and for an allowable unit stress of 15 000 pounds or 15 kips per square inch the flange area must be $1338.8/15 = 89.25$ square inches.

For the maximum negative moment the moments, when taken in the same order as above, are + 74.9, − 672.3, − 1079.7, and + 814.0 kip-feet, while the corresponding axial thrusts are 1460.7, 215.4, 41.5, and − 31.3 kips making the maximum compression in the lower flange 1015.8 kips.

Since at section 8 a flange area of 92.56 square inches is required the flange from that section to the crown will be given this area. Its composition is as follows:

6 Angles, 6″ × 6″ × 9/16″	38.58 sq. in.
3 Plates, 14″ × 9/16″	23.64
3 Plates, 18″ × 9/16″	30.36
Total area,	92.58 sq. in.

The arrangement of the shapes for the upper flange is shown in Fig. 122. The center of gravity of the flange is 4.66 inches above the backs of the lower angles, making the web plate about 50½ inches deep. If the web be spliced at every section each sheet will measure very nearly 52 inches out to out, since for this length the offset from the parabolic axis to its chord is nearly ¾ inch. The moment of inertia of the flanges is 170 070 inches^4.

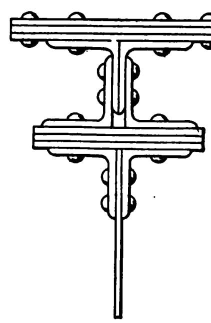

Fig. 122.

At section 1 the maximum flange stress occurs when the negative moment is a maximum, at 0 under the same load as for section 1, and at the remaining sections when the positive moment is a maximum. If the flanges are curved, as is usually the case, an additional moment must sometimes be considered when the maximum stress occurs in the lower flange. The line of action of the resultant of the thrust coincides with the chord of the axis in each division or panel and the moment due to this is a maximum at the middle of the division, the lever arm being the offset from the chord to the axis. In the present example the moment at the right end of each of the two end divisions is so much larger than at the middle that the effect of the curvature of the rib need not be computed. It is also found that the flange remains in compression throughout the arch under all conditions of loading.

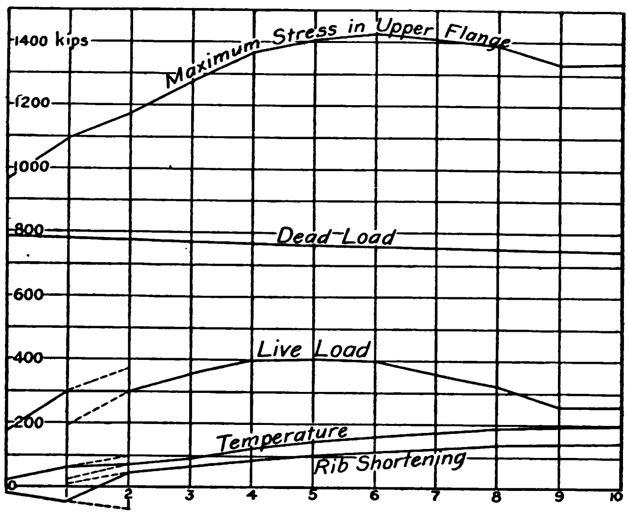

Fig. 123.

Fig. 123 shows the variation of the maximum stress for the left half of the arch. The greatest compression in the flange occurs at section 6 and the stress decreases toward the left and also somewhat toward the crown. At the crown the stress due to the dead and live loads is about 75 per cent of the total.

The diagram shows that the customary assumption that the moment of inertia increases from the crown to the hinges as the secant of the angle of inclination of the axis is not even approximately true outside of the quarter points. The stresses obtained are however sufficiently near to the true values as to answer the purposes of design in most cases.

Prob. 82. What should be the composition of the flanges on either side of section 6 provided the change from that at other sections is confined to the outer cover plates and that no plate is to exceed $\frac{9}{16}$ inch in thickness?

ART. 83. SHEAR.

In Art. 72 was given the method of finding the position of the live load which combines with other loads to make the shear a maximum in any given section. The method is exactly the same for a two-hinged arch in which the curved reaction locus replaces the rectilinear reaction locus of the three-hinged arch. Referring to Fig. 120, Art. 78, one point of division for the live load which causes the greatest shear in section 5 is found by drawing through the hinge at a a line parallel to the tangent to the axis at 5, and which intersects the reaction locus a little to the right of 13. As the section is supposed to be passed immediately on the left of 5 there is another point of division between 4 and 5 for the reasons given in Art. 63. Therefore the greatest positive shear is due to loads 5 to 13 inclusive, and the greatest negative shear to loads 1 to 4 and 14 to 19 inclusive.

In the arch rib under consideration the maximum shear at every section from 1 to 10 is positive. The most convenient method of finding the shear at any section, as for example, section 5, is to obtain the values of V_1 and H for the required loading, project each force on the normal section at 5 and add the results algebraically. The diagram in Fig. 120 does not show the true direction of the section since the ordinates are exaggerated.

If the dead load were uniformly distributed over the span there would be no normal shear at any section but as it is concentrated in panel loads the shear produced is the same as if there were no loads on the left half of the arch, while V_1 equals a half panel load or 29.5 kips, and $H = -2.9$ kips (see Art. 77). When the live loads 5–13 are on the arch $V_1 = 4.95 \times 18.2 = 90.1$ kips and $H = 15.8814 \times 18.2 = 289.1$ kips (see table in Art. 77). For positive shear the thrust H for temperature must have its largest negative value of 41.5 kips (Art. 81), which occurs under the specified decrease of 75 degrees in temperature. Under the given loading the average thrust in the arch is $1504.3 + 332.8 - 40.4 = 1796.7$ kips which makes $H = -31.8$. The total value of V_1 is therefore $29.5 + 90.1 = 119.6$ kips, and that of H is $-2.9 + 289.1 - 41.5 - 31.8 = 212.9$ kips. The sum of the components of these forces in the direction of the normal section at 5, as found graphically, is $+117.0 - 42.1 = +74.9$ kips.

The maximum shears at the sections 1 to 10 inclusive are respectively, +100.5, +90.0, +82.0, +70.2, +74.9, +76.7, +79.6, +81.0, +81.2, and +79.6 kips. They are preferably found for the sections at the points indicated rather than for intermediate ones since the web plate is spliced at those sections.

If the web plate be assumed as $\frac{1}{2}$ inch thick at section 1, and a bearing of 15 kips per square inch be allowed on the rivets, the bearing value of a $\frac{7}{8}$-inch rivet is 6.56 kips, and $100.5/6.56 = 16$ rivets will be required. This leaves a net section of $(50.5 - 16)\frac{1}{2} = 17.25$ square inches, provided holes 1 inch in diameter be deducted. If the allowable shearing stress in the web be 6 kips per square inch the net section required is $100.5/6 = 16.75$ square inches, indicating that the assumed thickness is correct. This thickness will be continued to section 3, a thickness of $\frac{7}{16}$ inch being used from there to the crown.

In order to find the number of rivets necessary to transmit the maximum increment of flange stress, for example, between sections 4 and 5, to the web plate, the flange stresses are found which are due to bending alone for the loading which produces the maximum shear in the web. The value of H is 212.9 kips as found above, but that of V_1 must be reduced by the half panel dead load, making it 90.1 kips for reasons that will be apparent on considering the statement at the end of Art. 78 respecting the moment due to dead load in connection with that relating to the dead-load shear in this article. The bending moment at 4 is $+90.1 \times 4 \times 12.9 - 212.9 \times 16.64 = 1106.5$ kip-feet, and that at 5 is $90.1 \times 5 \times 12.9 - 212.9 \times 19.50 = 1659.9$ kip-feet. As the effective depth is 5 feet, the difference of flange stress is $(1659.9 - 1106.5)/5 = 110.7$ kips. The bearing value of a $\frac{7}{8}$-inch rivet in a $\frac{7}{16}$-inch web being 5.74 kips for the allowable stress given above, the number of rivets required is $110.7/5.74 = 20$. As this number is so small the pitch would be reduced below that required on other considerations. Adequate provision must be made for the transfer of the panel loads from the posts to the web of the arch rib, and as this load is transmitted through some of the flange rivets to the web, their pitch near the post connection should be considerably less than elsewhere.

Prob. 83. Find the maximum shears, due to the dead and live load only, for sections 0 to 5 of the arch rib in Prob. 77.

ART. 84. ARCH RIB WITH OPEN WEB.

When the chords are united by open bracing as in Fig. 124 the maximum stresses in the chords may be found in the same manner as for a solid web by using as centers of moments the points where the verticals are intersected by the axis of the rib.

Since S_1 has its center at the lower panel point 5 its maximum stress is due to the loading which causes the maximum moment in section 5–5, while S_3 receives its maximum stress when the

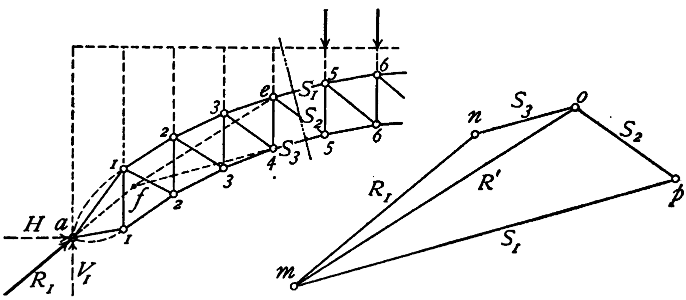

Fig. 124.

moment is a maximum in section 4–4. If the loading were found for the maximum moment at the lower panel point 5 and the value of the moment divided by the lever arm of S_1 the same result would be obtained as by adding the stress due to the maximum moment at the intersection of the axis and section 5–5, and that due to the corresponding axial thrust. It saves considerable labor to use the latter method.

To illustrate the method of finding the stress in any web member, as the diagonal S_2, let one of the chords cut by the given section, as S_3, be produced until it meets at f the line of action of the reaction R_1 for the required loading, and let this point be joined with e which is the intersection of the other two members cut by the section. If R_1 is the only external force on the left of the section it is held in equilibrium by the three stresses S_1, S_2 and S_3 and the line ef therefore is the line of action of the resultant R' of the stresses S_1 and S_2. By means of the force diagram let R_1 be resolved into S_3 and R' and then R' again resolved into S_1 and S_2. Since R_1 acts from m toward n, S_3 will act from n toward o, and hence toward the right or away from the section, which indicates tension. R' acts from o toward m, and as it is the resultant of S_1 and S_2, S_2 will act from o toward p and S_1 from p toward m. S_2 is therefore tension and S_1 compression. If any loads are on the left of the section the line of action of the resultant of R_1 and those loads must be found and then the resolution of the resultant made like that for R_1 above.

When S_1 and S_3 are nearly parallel as in Fig. 124, the stress in S_2 may also be found by obtaining the maximum shear in a section perpendicular to the mean of their directions or to the axis of the arch at the middle of the panel, in the manner explained in Art. 83. On multiplying the shear by the secant of the angle which S_2 makes with this section its stress is obtained.

It is desirable that the web members should consist of verticals and diagonals rather than that all of them should be inclined, because it was practically assumed in the deduction of the formula for H that the load is applied at the axis.

The graphic analysis employed for the three-hinged arch in Art. 66 may be used with equal advantage for two-hinged trussed arches. If by diagrams similar to Figs. 94 and 95 the stresses be found for V_1 and H equal to 1 kip each, the combined stress for any given values of the reactions may be obtained by taking the sum of two simple products. When this analysis is employed the reactions are preferably found by the method given in Art. 86, and illustrated by an example in Arts. 87–90.

Prob. 84. A parabolic arch rib has a span of 198 feet and a rise of 33 feet to a point midway between the chords. The chords are 6 feet apart and the bracing is like that shown in Fig. 124. The floor system is divided into 12 equal panels and the supporting bents rest on the ribs at its alternate panel points. Draw the stress diagram for a dead panel load of 64 kips, no load being applied at the panel points of the rib between the bents.

ART. 85. CIRCULAR ARCH RIB.

By introducing the equation of the circle instead of the parabola in Art. 77, the value of H may be found for a circular arch rib. The resulting integrations are however very tedious and the final expression for H contains so many terms that special tables are required for its convenient evaluation. If in the article just mentioned the axis of the arch used as an example were circular, the value of H would be 1.339 for a load of unity at panel point 5, instead of 1.381 for the parabolic axis. The circular axis of the arch would, however, lie wholly outside of the parabolic, and if the rise of the former were reduced so that both curves should include the same area above the line joining the hinges, the difference between the values of H would be considerably less.

When the rise is one-fifth of the span and a load of unity is placed at the quarter point the values of H are 0.669 and 0.696 for the circular and parabolic arches respectively. If the rise is one-fourth of the span and the load is placed in the same position the corresponding values of H are 0.562 and 0.556. It is observed that for this ratio of rise to span the H of the circular arch now slightly exceeds that of the parabolic.

Very few arches have been built with less than three hinges whose rise exceeds one-fourth of the span and within this limit the approximate stresses in a circular arch may be found by substituting for its reactions those of a parabolic arch whose axis encloses the same area.

Prob. 85. Compute the rise of a circular arch rib whose span is 210 feet and whose axis shall enclose the same area as the axis of a parabolic arch of the same span and a rise of 30 feet. Compare the ordinates at intervals of one-tenth of the span. Which form has the advantage on account of æsthetic considerations?

ART. 86. REACTIONS FOR A BRACED ARCH.

The term 'braced arch' is sometimes applied to the case where two curved chords are connected by bracing, as in Fig. 124, but more commonly to the case where the structure consists of an arched lower chord and a horizontal upper chord as in Figs. 125 and 127. The latter form is generally called the spandrel-braced arch. While an arch rib is analogous to a beam, a braced arch is analogous to a truss and hence the horizontal thrust H cannot be derived by the formula established in Art. 76 from the consideration of pure flexural stresses. It is now proposed to deduce a method of determining H for the braced arch with two hinges.

Let Fig. 125 represent any braced arch whose span is l and let a load P be placed at any distance kl from the left end.

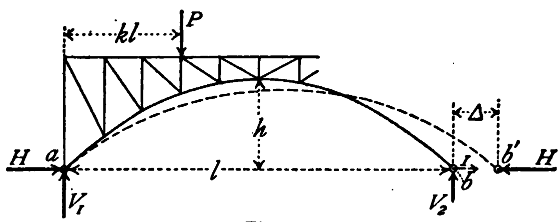

Fig. 125.

The reactions at the supports are resolved into their vertical and horizontal components, and, as for the arch rib,

$$V_1 = P(1 - k), \qquad V_2 = Pk, \qquad (1)$$

are the vertical components of the reactions.

To find the horizontal component, or horizontal thrust H, similar reasoning to that in Art. 76 will be employed, the hinge b being supposed to move horizontally to b' under the action of P and then brought back to b under the action of H. Thus the displacement Δ due to P is equal to that due to H, or H must have such a value that the length of the span remains unchanged. The method of expressing the two values of the displacement Δ is similar to those given in Arts. 78 and 79 of Part I for finding the deflection of a point in a truss.

Let L be the length of any member of the trussed structure and A the area of its cross-section. Let S' be the stress produced in it by P when the end b is free to move. Then the elongation of that member is $S'L/AE$. Now suppose a horizontal force of unity to act at b in the direction bb' and let T be the stress in the member produced by it. The internal work in the member then is $\frac{1}{2}S'TL/AE$, and this resists the external work $\frac{1}{2}(1 \times \Delta)$. Hence taking all the members of the structure,

$$\Delta = \Sigma \frac{S'TL}{AE} \qquad (2)$$

is the horizontal displacement produced by the load P.

Again, let U be the stress in any member produced by the thrust H, or $U = HT$. Then by the same reasoning,

$$\Delta = \frac{1}{H}\Sigma\frac{U^2L}{AE} = H\Sigma\frac{T^2L}{AE} \qquad (3)$$

is the horizontal displacement produced by the thrust H.

Equating these two values of Δ gives the general formula

$$H = \Sigma\frac{S'TL}{AE} \Big/ \Sigma\frac{T^2L}{AE}, \qquad (4)$$

which is an expression for computing the horizontal thrust for any trussed two-hinged arch due to a load P. It may also be noted that this formula is an expression for the horizontal thrust due to any system of loading provided that S' represents the stresses due to that loading. In all cases the stresses S' are computed from the vertical reactions V_1 and V_2 exactly as if the structure were a simple truss.

As an elementary illustration let a straight beam of uniform section be bent so as to form two straight rafters having a span

l and a rise h, the bend being at a horizontal distance $\frac{2}{3}l$ from the left end and $\frac{1}{3}l$ from the right end. Let these ends be provided with hinges and a load P be applied at the peak. Then $V_1 = \frac{1}{3}P$ and $V_2 = \frac{2}{3}P$, which produce in the two struts, whose lengths are L_1 and L_2, the two stresses,

$$S'_1 = \frac{PL_1}{3h}, \qquad S'_2 = \frac{2PL_2}{3h},$$

while a horizontal force unity at the hinge gives the stresses,

$$T_1 = \frac{3L_1}{2l}, \qquad T_2 = \frac{3L_2}{l}.$$

As the values of A_1 and A_2 are equal and E is the same for both rafters, the formula (4) becomes,

$$H = \frac{S'_1T_1L_1 + S'_2T_2L_2}{T_1^2L_1 + T_2^2L_2} = \frac{2Pl}{9h},$$

which is the horizontal thrust due to the load P.

Prob. 86. A straight beam of uniform cross-section is bent at the middle so as to have the span l and rise h, the two halves remaining straight. Hinges are placed at the ends, one being fixed in position and the other free to move horizontally. Compute from (2) the horizontal displacement of the free end when a load P is applied at the peak.

ART. 87. THE SPANDREL-BRACED ARCH.

The usual form of the spandrel-braced arch is shown in Fig. 126. The vertical reactions V_1 and V_2 are the same as for a simple truss. The horizontal thrust H, however, depends upon the lengths and cross-sections of the members as shown by (4) of Art. 86. When V_1, V_2, and H have been found for any given load P the stresses due to that load are readily determined by the methods of Chapter V. Let S' be the stress in any member when the arch is treated as a simple truss, and let T be the stress in the same member due to a horizontal force of unity at the hinge acting in the direction of H. Then

$$S = S' + HT \qquad (1)$$

gives the stress in that member under the action of all the forces P, V_1, V_2, and H. The stresses S' and T used in computing H may hence be directly used to find the final stress S.

In order to determine H by the graphic method it will be best to modify formula (4) of Art. 86. For this purpose let δ be the deflection of the arch under the load P due to a horizontal force unity acting at the hinge and away from it; whence the deflection under the load due to a horizontal force P acting at the hinge is $P\delta$. Then the horizontal displacement $\Delta = P\delta$, because in (4) it is immaterial whether S' be the stress due to the vertical load P and T that due to a horizontal force unity at the hinge, or whether S' be the stress due to a horizontal force P at the hinge and T that due to a vertical load of unity. Further, let δ' be the horizontal displacement of the hinge due to a horizontal thrust of unity, then $H\delta'$ is the displacement due to H, or Δ equals $H\delta'$. Accordingly

$$H\delta' = P\delta \text{ or } H = P\frac{\delta}{\delta'} \qquad (2)$$

is the formula for use in determining H by the graphic method, δ and δ' being found by a displacement diagram.

If the areas of the cross-sections are known for all the members of a braced arch the elongation λ may be computed for each member due to the stress produced by a horizontal thrust of unity, and a displacement diagram similar to Fig. 105 be constructed in the manner described in Chapter VII of Part II. As the truss is symmetrical only one-half of the diagram need be drawn. From this diagram the horizontal displacement δ' at the hinge and the vertical deflection δ at each panel point may be directly measured, and then from (2), H is found for each panel load. As the ratio of δ to δ' depends only on their relative values the computations may be abbreviated by taking E equal to some convenient unit, as 1 or 1000.

Very close approximate values of H may be found by assuming all cross-sections to be the same and for the reason given above this may also be taken equal to unity. For example, let the truss of the spandel-braced arch erected at Niagara Falls

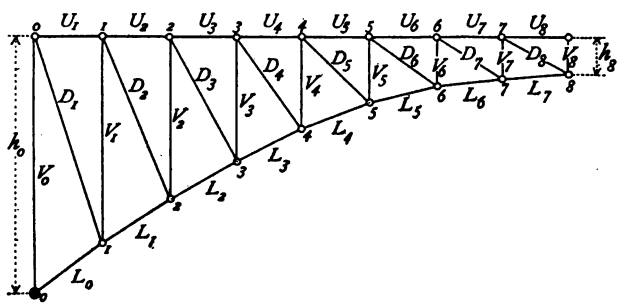

Fig. 126.

in 1897 be taken. The span is 550 feet, and the rise of the lower chord in the plane of the truss is 113.9 feet. Each upper chord member is 34.375 feet long, and the lengths of the lower chord members, beginning at the end, are 43.621, 41.511, 39.616, 37.964, 36.588, 35.521, 34.792, and 34.422 feet respectively. The lengths of the verticals are 134.000, 107.817, 84.546, 64.854, 48.742, 36.211, 27.261, 21.896, and 20.100 feet, while those of the diagonals are 113.164, 91.267, 73.403, 59.644, 49.929, 43.873, 40.753, and 39.820 feet respectively. These dimensions, the dead panel loads, and the data given in the first paragraph of Art. 90, were furnished by L. L. BUCK, Chief Engineer.

The stresses due to a horizontal thrust of unity are given in the following table. Since these values are also to be used later as multipliers for the ordinates of the influence diagrams in determining the live load stresses as well as in finding the temperature stresses, it is best to obtain them by computation.

STRESSES DUE TO $H = 1$. (SEE FIG. 126.)

SUBSCRIPTS.	U	L	D	V
0	—	−1.257	—	+0.762
1	+0.243	−1.501	−0.799	+0.841
2	+0.585	−1.827	−0.908	+0.908
3	+1.066	−2.282	−1.028	+0.968
4	+1.749	−2.926	−1.185	+1.002
5	+2.701	−3.824	−1.382	+0.964
6	+3.915	−4.975	−1.551	+0.768
7	+5.121	−6.130	−1.430	+0.319
8	+5.667	—	−0.631	0

By expressing the lengths of the members in inches, the stresses in kips, and taking E as 1000 kips per square inch and the area of cross-section A as one square inch for all the members, the computed values of λ will be expressed in inches. By drawing a displacement diagram to a scale of one eighth, the following deflections δ were found for the upper panel points 0 to 8 inclusive: 1.22, 27.66, 53.1, 77.4, 100.4, 121.4, 139.3, 152.0, and 156.38 inches. The horizontal displacement δ' of the right hinge b was found to be $2 \times 88.4 = 176.8$ inches. On dividing each deflection by δ' the following reactions H are obtained for a load unity placed successively at the upper panel points 0 to 8 inclusive: 0.007, 0.156, 0.300, 0.438, 0.568, 0.687, 0.788, 0.860, and 0.885. When these values are laid off as ordinates in Fig. 128 the influence line for H is obtained. It will be shown in Art. 90 that the final value of H for a load at the middle differs from this approximate value by about four per cent.

The dead panel loads 0 to 8 inclusive are 750, 475, 399, 366, 330, 324, 320, 329, and 319 kips respectively. The large panel load at the end is due to the fact that the arch supports one end of an approach span 115 feet long. To increase the lateral stability the plane containing the center lines of the truss members is inclined, so as to make an angle with the vertical whose tangent is one-tenth. The panel loads in the plane of the truss are therefore the product of the vertical panel loads and the secant of this angle, and are found to be 753.7, 477.4, 401.0, 367.8, 331.6, 325.6, 321.6, 330.6, and 320.6 kips. The value of H due to the dead load is found by multiplying each panel load by the corresponding ordinate in Fig. 128 and adding the products, the result being 2905.4 kips. $V_1 = V_2 = 3469.6$ kips. The dead load stresses may now be found by constructing a stress diagram. The stress in L_8 is -2652; in U_8, -2115; in D_4, $+420$; and in V_3, -711 kips. Since the lower panel points do not all lie in a parabola, and the panel loads are not equal, there are dead load stresses in all the web members, those in the diagonals and in the verticals respectively varying but little comparatively throughout the span. The stresses in both chords are compression throughout, those in the upper chord increasing toward the middle and those in the lower chord from the middle toward the ends.

Prob. 87. Construct the displacement diagram referred to above, to determine the vertical deflections of the upper panel points and the horizontal displacement of the right hinge.

ART. 88. LIVE LOAD STRESSES.

The advantages of influence lines in determining the live load stresses are so great in this case that they will be employed. Let M' be the bending moment for any given truss member due to the vertical load and the vertical components of the corresponding reactions, that is, as if the arch acted as a simple truss; and let y be the ordinate to the given center of moments measured from the axis through the end hinges. Then the required bending moment is $M = M' - Hy$, and if r is the lever arm of the member, its stress is $S = \frac{M}{r} = \frac{1}{r}(M' - Hy)$. In order that the influence line for H may be employed directly let this equation be transformed into

$$S = \frac{y}{r}\left(\frac{M'}{y} - H\right). \qquad (1)$$

The greatest tension and compression in the member due to the live load may therefore be found by constructing the influence line of M'/y in such a relation to that of H, that the ordinates between them will represent the difference between their respective ordinates, and after adding the positive and negative ordinates separately multiplying the respective sums by the quantity y/r, which is known as the multiplier of the influence diagram.

To aid in avoiding errors it is desirable to lay off the ordinates of both influence lines in such a way that the ordinates between the lines shall indicate tension when above the H-line and compression when below it. With this arrangement the multiplier in equation (1) will always be regarded as positive.

For example, let it be required to find the live load stresses in L_3 of Fig. 127. Its center of moments is at panel point e_3. For a load unity on the right of e_3 the left reaction V_1 is $1 \cdot x'/l$ if x' be the distance from the load to the right support. Let x be the distance from the left support to the center of moments, then $M' = x'x/l$, and since y in this example equals h_0, the depth of the truss at the end, $M'/y = x'x/h_0 l$. If this relation were also true when the load is on the left of the center of moments the value of M'/y when the load is at the left end would be x/h_0, which for L_3 becomes $3p/h_0$, p being the panel length. Substituting the numerical values of p and h_0, this quantity equals $3 \times 34.375/134.0 = 0.770$. Let this value be laid off as an ordinate $a'a_3''$ in Fig. 128, and let the right line connecting

a_8'' and b' intersect the vertical through the center of moments e_3 at e_3', then $a'e_3'b'$ is the line of influence of M'/y for L_3.

The direction in which $a'a_3''$ should be laid off is most readily determined by observing that the product of x/h_0 and the multiplier in equation (1) is the stress in L_3 due only to a vertical reaction of unity at a, which is tension (+), and hence the ordinate is laid off above the axis $a'b'$. The stress in L_3 due to H is the product of H and the multiplier and this is compression (−), but since this is the subtraction term in equation (1) it becomes +, and therefore the ordinates of the H-line are also laid off in a positive or upward direction from the axis.

Fig. 128 shows that the greatest tension is due to the live panel loads 1 to 4, and the greatest compression due to the loads 0 and 5–16 inclusive. Since the loads at 0 and 16 are equal respectively to the reaction of one approach span plus a half panel load on the arch, the stresses due to them will be determined separately.

The specified live load for each track consists of a uniform load of 3500 pounds per linear foot preceded by two locomotives each weighing 256 000 pounds and 54.25 feet long, and for the highway floor a uniform load of 3000 pounds per linear foot. The uniform panel load per truss is therefore 171.9 kips and for convenience let the excess of the locomotives over the corresponding uniform load be represented approximately by the addition of four-tenths of a panel load to the first two panel loads. When reduced to the plane of the truss the live panel load is 172.8 kips. On account of the two locomotive panel loads the greatest tension is obtained in L_3 by placing them at 3 and 2 and leaving 4 unloaded. The ordinate at 1 measures 0.053 and the sum of those at 2 and 3 is 0.304, therefore the stress in L_3 is

$$S = +(0.053 + 0.304 \times 1.4)172.8 \times 2.282 = +189 \text{ kips.}$$

The quantity 2.282 is the multiplier y/r in equation (1) and is numerically equal to the stress in L_8 due to a horizontal thrust of unity. See table in Art. 87. For the reason previously given no attention need to be paid to its sign in the preceding

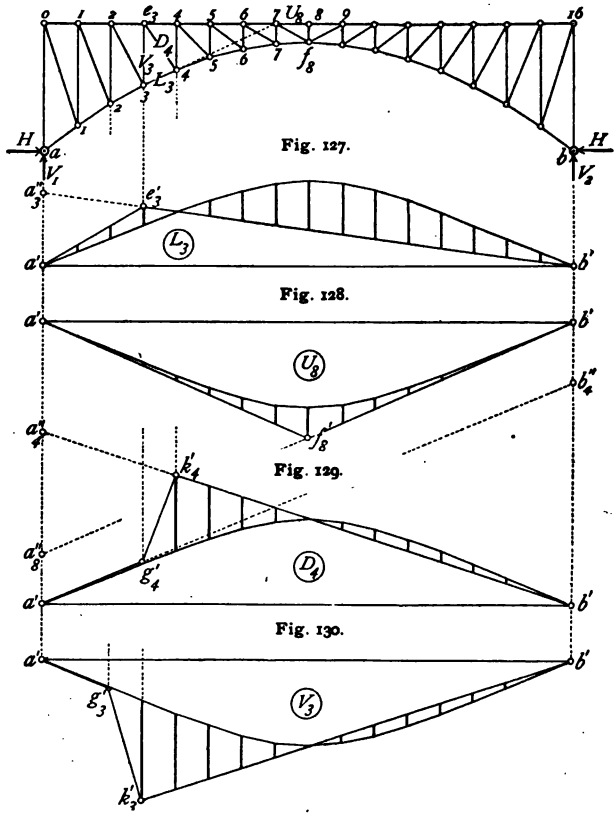

Fig. 127.

Fig. 128.

Fig. 129.

Fig. 130.

Fig. 131.

equation. The locomotive panel loads will next be placed at 5 and 6 and the remaining loads at 7 to 15 inclusive. The sum of the ordinates at 5 and 6 is -0.464 and that of the rest is -3.376, making the stress

$$S = -(0.464 \times 1.4 + 3.376)172.8 \times 2.282 = -1587 \text{ kips.}$$

The influence lines of M'/y for all the lower chord members may be placed in the same diagram, Fig. 128, provided care be exercised in adding the ordinates with the dividers. The points a'' will be equidistant and the points e' will lie on a parabola.

In Fig. 129 the corresponding influence lines for U_8 are given. The ordinates to the straight lines are laid off below the axis because the stress in U_8 due to a vertical reaction of unity is compression $(-)$. The stress due to the horizontal thrust of unity is tension $(+)$, which sign becomes minus on account of the sign of H in equation (1), and hence the ordinates of the H-line are also laid off downward. The ordinate $a'a_8''$ equals

$$\frac{M'}{y}=\frac{8p}{y}=\frac{8\times 34.375}{113.9}=2.414.$$

The original diagram showed that the loads 2–14 cause compression in U_8. Since the sum of the ordinates at 2 and 3 is however less than four-tenths of that at 5, the largest compression is obtained by placing the locomotive panel loads at 4 and 5, and the other panel loads at 6 to 14. Since the multiplier is 5.667 the stress in U_8 is

$$S=-(0.103\times 1.4+1.068)172.8\times 5.667=-1187 \text{ kips}.$$

For the tension in U_8 a locomotive panel load is placed at 1 and a train panel load at 15. As each of the corresponding ordinates measure 0.005, the stress is $+12$ kips.

If the panel points of the lower chord were all on a parabola the points f' for all the upper chord members would lie on a straight line parallel to the axis $a'b'$.

The influence lines for D_4 are shown in Fig. 130. The center of moments is at the intersection of the chord L_3 with the horizontal upper chord, which is 0.87 foot on the right of panel point 7, making $x=241.49$ feet and $M'/y=x/h_0=241.49/134.0=1.802$, which is laid off as the ordinate $a'a_4''$. The ordinate is positive since the vertical reaction of unity at a causes tension in D_4. The ordinate $b'b_4''$ equals x'/h_0, x' being the distance from the center of moments to the right support. The product of x'/h_0 and the multiplier of the influence diagram equals the stress in D_4 when the truss is regarded as fixed at the left end and subject only to the vertical reaction of unity at b, and hence the ordinate must also be laid off upward. Its value is $(550 - 241.49)/134.0 = 2.302$. A useful check on the construction is that the lines $a'b_4''$ and $a_4''b'$ must intersect in the vertical through the center of moments of D_4. As shown in Art. 74 the influence line for M'/y for this diagonal is $a'g_4'k_4'b'$, the right line $g_4'k_4'$ being located in the same panel of the loaded chord as is cut by the section used in finding the stress in D_4 by the method of moments. When drawn to a large scale Fig. 130 shows that the greatest compression is due to the loads 1–3 and 9–15 inclusive. Placing the locomotive panel loads at 9 and 10, the stress is found to be

$$S = -(0.029 + 0.184 \times 1.4 + 0.460)172.8 \times 1.185 = -153 \text{ kips.}$$

The loads at 1, 2 and 3 may be regarded as due to the rear of a preceding train. The greatest tension is due to panel loads 4–8 inclusive, the locomotive panel loads being placed at 4 and 5, and the other panel loads at 6, 7, and 8. The result is

$$S = +(1.335 \times 1.4 + 0.508)172.8 \times 1.185 = +487 \text{ kips.}$$

Since the vertical V_3 has the same center of moments as D_4, the influence lines for its M'/y will be the same as for D_4 except that the right line $g_3'k_3'$ is moved a panel toward the left. This is due to the fact that a section cutting V_3 and two of the chords passes between the panel points 2 and 3 of the loaded chord. The greatest live load tension is

$$S = +(0.024 + 0.184 \times 1.4 + 0.460)172.8 \times 0.968 = +124 \text{ kips,}$$

and the greatest compression is

$$S = -(1.810 \times 1.4 + 1.060)172.8 \times 0.968 = -601 \text{ kips.}$$

For V_0 the influence diagram for M'/y is a right triangle, the point g_0' coinciding with a', and k' with a_0'' (compare Fig. 131). The influence lines and stresses for the remaining truss members are found in the manner described above, those for each set of members, as the upper chords, lower chords, diagonals and verticals, being combined in a single diagram.

The approach trusses have 6 panels and a span of 115 feet (see Engineering News, Aug. 6, 1896). A full uniform live load will therefore cause a reaction of $\frac{1}{2} \times 115 \times 5.0 = 287.5$ kips. When reduced to the plane of the arch truss and a half panel load is added, the live panel load at 0 and 16 becomes 375.4 kips. With the aid of the table in Art. 87 the stresses due to both loads 0 and 16 are found to be: -24 kips in L_3; $+60$ kips in U_8; -12 kips in D_4; and $+10$ kips in V_3. Since a locomotive panel load instead of a train panel load must be placed at 0 (one having previously been placed at 1) to obtain the greatest tension in U_8 it is necessary to add the stress of $+2$ kips to the preceding one. These live load stresses must then be added to those of the same kind previously found for the members L_3, U_8; D_4, and V_3.

In case it be desired to find the stresses due to specified wheel loads it may be done as explained in Art. 74. For trusses where the live load stresses are relatively so small as in the example under consideration this is regarded as an unnecessary refinement.

Prob. 88. Draw the influence lines for L_0, V_0 and D_8 in Fig. 126, and find their greatest tension and compression due to the live load.

ART. 89. TEMPERATURE STRESSES.

In Art. 87 it was shown that the horizontal displacement of the right support b due to a horizontal thrust of one kip, and with the assumed values for E of 1000 kips per square inch and for cross-section areas A of one square inch, was 176.8 inches. If E is 29 000 kips per square inch the values of the elongations λ were laid off $29A$ times their true value and hence the corresponding displacement in inches is $176.8/29A$. For a change in temperature of 75 degrees and a coefficient of expansion of 0.0000065, the span being $550 \times 12 = 6600$ inches, the change in the length of the span is

$$0.0000065 \times 75 \times 6600 = 3.218 \text{ inches,}$$

and hence the corresponding horizontal reaction is

$$H = 3.218 \times 29A/176.8 = 0.528A \text{ kips.} \qquad (1)$$

The value of A in this equation is not a simple average of the areas of cross-section of the truss members but a weighted mean, the weight of some members being much greater than that of others. For instance, on modifying the displacement diagram, referred to in Art. 87, so as to omit the elongations of all the web members the value of H for a load at the middle was reduced only $2\frac{1}{4}$ per cent, the difference increasing to about 3 per cent for a load at panel points 2 and 3. The influence of the chord members on H increases very rapidly toward the middle since the lever arm of H increases while at the same time the lever arm of the stress decreases. An examination of the displacement diagram also shows that a given elongation or shortening of a chord near the middle of the span causes a much larger deflection than if it occurred in a chord member toward the ends of the span. On account of these considerations the mean value A will not differ very far from the area of U_8 whose influence on H is the greatest of all the members.

In order that the approximate temperature stresses may be on the safe side let A in equation (1) be assumed as equal to that of U_8. The stress in U_8 due to a horizontal thrust of one kip is 5.667 kips and hence the temperature stress equals

$$S = 5.667 \times 0.528A = 2.99A \text{ kips,}$$

which is tension for a rise and compression for a fall in temperature. The dead load stress is -2115 kips, and that due to live load is -1187 kips, and to these must be added the wind stresses. The cross-section area of U_8 can then be readily found by trial for any specified unit stress per square inch. The final area is 346.67 square inches, and therefore the temperature stress in U_8 is $\pm 2.99 \times 346.67 = \pm 1037$ kips. The corresponding value of H is obtained by equation (1) and is

$$H = 0.528 \times 346.67 = 183.0 \text{ kips.}$$

The preliminary temperature stresses in all the members may now be obtained by multiplying the stresses in the table in Art. 87 by this value of H.

ART. 90. FINAL HORIZONTAL REACTIONS.

The areas of cross-section adopted in the design of the arch which was used as an example in Arts. 87–89 are as follows, being expressed in square inches: In the upper chord beginning at the end, 139.0, 139.0, 159.25, 215.59, 276.01, 334.46, 352.21, and 346.67; in the lower chord, 633.34, 579.09, 564.09, 540.09, 524.84, 498.52, 457.52, and 382.19; in the diagonals, 165.17, 156.96, 142.0, 131.88, 112.88, 152.0, 221.0 and 295.37; and in the verticals 254.78 (246.76), 194.22, 182.22, 169.22, 157.34, 149.34, 149.34, 150.34 and 83.84.

The lengths of the members and their stresses due to a horizontal thrust of one kip are given in Art. 87. The elongations are next computed by dividing the corresponding values used in that article by the respective areas. By means of a displacement diagram, drawn to a scale of 0.02 inch to an inch, the deflections of the upper panel points 0 to 8 were found to be as follows: 0.0049, 0.0933, 0.1777, 0.2574, 0.3318, 0.3998, 0.4553, 0.4933 and 0.5069 inches. The horizontal displacement of the right support was found to be $0.2755 \times 2 = 0.551$ inch.

On dividing the deflection of each panel point by the horizontal displacement of the support the horizontal reaction H is obtained due to a vertical load of one unit applied at that point. If the load be 1 kip, H will also be expressed in kips. The values obtained are as follows: 0.009, 0.169, 0.323, 0.467, 0.602, 0.726, 0.826, 0.895, and 0.920. The preliminary value of H for a load unity at the middle was 0.885 which is nearly 4 per cent less than the final value. The greatest difference is that for a load at 5 and amounts to 0.039; from that point to the end the difference decreases in magnitude while it increases in proportion to the corresponding values of H.

The revised value of H for the dead load is 3067.8 kips and the dead load stresses, also expressed in kips, are given in the following table:

DEAD LOAD STRESSES. (SEE FIG. 126.)

SUBSCRIPTS.	U	L	D	V
0	—	−3855	—	−1132
1	− 121	−3559	+397	− 721
2	− 222	−3280	+264	− 606
3	− 331	−3021	+233	− 554
4	− 464	−2772	+228	− 519
5	− 641	−2507	+259	− 506
6	− 868	−2226	+290	− 470
7	−1097	−1973	+272	− 386
8	−1194	—	+114	− 321

The final value of H due to temperature is $3.218 \times 29/0.551 = 169.3$ kips (see Art. 89). The revised stresses in L_3, U_8, D_4, and V_3 are as follows:

	L_3	U_8	D_4	V_3
Dead load	−3021	−1194	+228	−554
Live load	+ 155	+ 196	+447	+192
	−1775	− 877	−235	−565
Temperature	∓ 386	± 960	∓201	±164

To obtain the revised live load stresses the only change required in the influence diagrams is that of the influence lines for H.

On comparing these dead and live load stresses with their preliminary values it may be seen that the relative difference between the corresponding values of H is no measure of the relative difference between the stresses.

The attention of the student is directed to the great advantage of arranging the computations in a systematic order and of preserving all the intermediate results as well as the logarithms employed in obtaining them, since so many of them are used several times in an example like that considered in Arts. 87–90.

Prob. 89. Find the deflection of the crown of the arch in the preceding example under full live load, and compute the range of its deflection caused by a change in temperature of ± 75 degrees Fahrenheit.

Prob. 90. Determine the final horizontal thrusts for the arch in Art. 73 provided it be made continuous at the crown.

ART. 91. EFFECT OF YIELDING SUPPORTS.

In the preceding article it was found that when H is equal to one kip and E is assumed as 1000 kips per square inch, the horizontal displacement of the right support is 0.551 inch. For $E = 29\,000$ kips a displacement of one inch corresponds to $H = 1.0 \times 29/0.551 = 52.6$ kips, and to a stress in U_8 of $52.6 \times 5.667 = 298$ kips. That is, if one of the supports of the arch should yield so as to increase the distance between the end hinges by one inch, the compression in U_8 (see Fig. 127) would be increased by 298 000 pounds, and for other displacements in the same proportion. This fact indicates the necessity of having unyielding foundations in order that a two-hinged arch may be adopted with safety and economy at a given locality. It also shows the importance of an accurate location of the hinges in the erection of the structure, so that the span may have the assumed value at the standard temperature.

A yielding of one inch horizontally of one of the supports of the two-hinged arch rib, used as an example in Arts. 77–83, would reduce the reaction H due to the dead and live loads by 27.55 kips and increase the positive bending moment at the crown by $27.55 \times 26.0 = 715.4$ kip-feet, which exceeds the bending moment due to the live load. This value of H is computed by making Δ equal to one inch in equation (3) of Art. 76 and proceeding in the same manner as in Art. 81.

Prob. 91. Construct the influence line for the reaction of one of the supports of the Niagara railroad arch whose dimensions and loads are given in Arts. 87, 88 and 90, and find its magnitude and direction when the reaction is a maximum.

ART. 92. THE CRESCENT BRACED ARCH.

Several metallic arches of crescent shape and with open web bracing have been constructed in Europe in places where high bridges were required, the live load being transferred to the arch at comparatively few points. Several of these structures were referred to in Art. 75. See the article on the St. Lawrence bridge competition in Engineering News, Jan. 7, 1897, and note the comment on the crescent braced arch.

The methods outlined in Arts. 87–90 for the spandrel-braced arch apply also to this type, or indeed to any form of trussed arch with two hinges. The curves of the chords are generally made parabolic, but the method of treatment is independent of the form of the curves, and the difference in labor on that account is comparatively slight.

Prob. 92. Refer to the Engineering News, Aug. 9 and 30, 1884, and notice the relation between the unit stresses in the Garabit viaduct due to the dead, live, and wind loads.

CHAPTER VII.

ARCHES WITHOUT HINGES.

ART. 93. DESCRIPTIVE NOTES.

When metallic arches with no hinges have their ends bolted to the abutment so that no change in direction can occur, bending moments are developed at the ends. Such arches are stiff and have high stresses due to temperature.

The finest example of the metallic arch with fixed ends is the St. Louis bridge over the Mississippi river, completed in 1874. The central arch has a clear span of 520 feet and a rise of about 47 feet, while the adjacent arches have a span of 502 feet and a rise of $43\frac{3}{4}$ feet. The arch ribs have chords 12 feet apart united by isosceles bracing (see Fig. 135). The four ribs in each span support a double-track railroad and a paved highway. Each track passes between two ribs at the crown. The load is transferred to the ribs at the panel points by vertical posts, there being 44 panels in the central span. The radius of the upper chord of the rib is 742 feet for all the spans. A full account of the construction and erection of this structure together with the tests of materials and the theory of the ribbed arch without hinges may be found in C. M. WOODWARD's History of the St. Louis Bridge, 1881. The volume contains 47 plates of detail drawings and photographic views.

The general details of a light military bridge over the Cerveyrette gorge in France are published in Engineering News, June 9, 1892. The span is 172.2 feet and the rise 37.72 feet. The arches have a depth of 2.42 feet at the crown and

6.56 feet at the springing line. This arch illustrates a type of which quite a number have been built in Europe. The arch is continuous throughout, the ends of the chords being held in position by shoes which receive their thrust. The ends are, however, not fixed by being bolted to the supports.

The Kaiser Wilhelm bridge over the Wupper river between Remscheid and Solingen in Germany, completed in 1897, has the longest span of any arch with fixed ends in Europe. Its span is 526 feet and the rise 223 feet. A skeleton diagram and some notes regarding the erection of the bridge are given in the Engineering Record, Dec. 25, 1897.

The Cornhouse bridge at Berne, Switzerland, completed in 1898, is noted for its beauty in form and proportion. The span of the main arch is 358 feet and the rise from the springing line to the center of the lower chord is 110 feet. The depth is less at the crown than at the springings. The floor and its load is transferred by eleven bents to the arch, these bents being placed at every third panel point. The bracing consists of radials and counter diagonals. See Engineering News, Dec. 19, 1895, and Dec. 16, 1897.

Prob. 93. Consult the engineering periodicals and make a list of all the metallic arches without hinges whose principal dimensions can be found. Record the span, rise, and ratio of rise to span, and copy the skeleton diagram of each arch.

ART. 94. CONDITIONS OF EQUILIBRIUM.

Let an arch rib with fixed ends have a span l and rise h and be subject to a load P at a distance kl from the left end. Since the ends are fixed the reactions R_1 and R_2 produce moments at the supports as in a beam whose ends are rigidly fastened. Consequently the lines of action of the reactions do not pass through the ends but cut the verticals drawn through those points either above or below the ends of the span. The left reaction R_1 has the components V_1 and H, and the right reaction

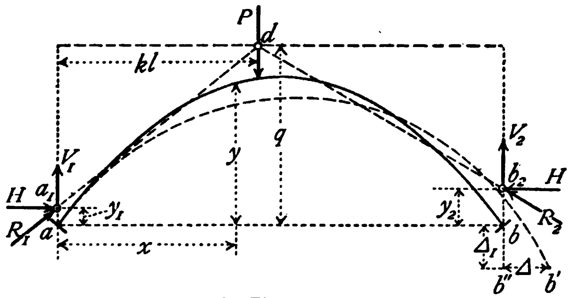

Fig. 132.

R_2 the components V_2 and H. The load P is thus held in equilibrium by the vertical forces V_1 and V_2 and the horizontal forces H and H.

Let y_1 be the height of H above a and y_2 the height of H above b. Then, taking moments about a,

$$Pkl - V_2 l - Hy_2 + Hy_1 = 0 \qquad (1)$$

is the static condition that rotation shall not occur. Also,

$$V_1 + V_2 - P = 0 \qquad (2)$$

is the static condition that upward or downward motion shall not occur. These equations contain five unknown quantities, V_1, V_2, H, y_1 and y_2, and hence three additional conditions are required to determine them.

The word 'fixed' means that the tangents to the arch rib at a and b do not alter their direction when the arch is deformed under the action of the load P. This gives a third condition. To develope it, consider an element ds of an arch rib whose ends were originally parallel, but which in consequence of the flexure have become inclined to each other at the angle $d\phi$. The angle $d\phi$ thus measures the change in the direction of the tangents at the two ends of ds. In Fig. 117 it is seen that pq represents

$cd\phi$. But pq also represents the elongation of the outer fiber under the actual bending moment M produced by all the exterior forces; this elongation is $\frac{Mc}{I} \cdot \frac{ds}{E}$. Equating these two values of pq gives the value of $d\phi$, whence

$$\phi = \int_0^l \frac{Mds}{EI}$$

is the total change in the angle between the tangents to the arch at the ends of the span. But if the ends remain fixed, this change must be zero. Accordingly

$$\int_0^l \frac{Mds}{EI} = 0 \qquad (3)$$

is the third condition of equilibrium for the arch with fixed ends.

The two remaining conditions are established from the consideration that the thrust H must be of such intensity and be applied at such heights y_1 and y_2 as to prevent the horizontal and vertical displacements $b'b''$ and bb'' which might be caused by P. By reasoning like that of Art. 76 it is seen that the horizontal displacement Δ may be expressed in terms of the moment M' due to the vertical forces V_1, V_2, and P, or in terms of the moment M'' due to the horizontal thrust H. Thus, y being the bending moment due to a horizontal force unity applied at b, and the two displacements being equal and opposite, there results

$$\int \frac{M'yds}{EI} + \int \frac{M''yds}{EI} = 0. \qquad (4)$$

Again, let a vertical force of unity be applied at b, the vertical displacement Δ_1 caused by P must be equal and opposite to that caused by H. Or, since x is the moment due to this force unity,

$$\int \frac{M'xds}{EI} + \int \frac{M''xds}{EI} = 0. \qquad (5)$$

These equations express the fourth and fifth conditions of equilibrium for the arch with fixed ends.

These five conditions enable the five quantities V_1, V_2, H, y_1, and y_2 to be deduced, and thus all the external forces acting upon the arch rib are known in intensity and position. From these the shears and bending moments in any section may be computed.

Prob. 94. An arch rib of 100 feet span and 20 feet rise has a load P at 25 feet from the left end. If H is equal to $\frac{1}{2}P$, and y_1 and y_2 are -14 and $+6$ feet respectively, find the values of V_1 and V_2.

ART. 95. THE PARABOLIC ARCH RIB.

Let a symmetrical parabolic arch rib with fixed ends have the span l and the rise h; the equation of its axis referred to an origin at the left end is $y = 4h(lx - x^2)/l^2$. To derive the reactions due to a load P situated at the distance kl from the left end, the same simplifications will be made as in Art. 77, namely, that E is constant, $ds = dx \sec i$, $I = I_c \sec i$. Then the equations (3), (4), (5) of the last article become, since $M = M' + M''$,

$$\int_0^{kl} M'dx + \int_{kl}^{l} M'dx + \int_0^{l} M''dx = 0,$$

$$\int_0^{kl} M'ydx + \int_{kl}^{l} M'ydx + \int_0^{l} M''ydx = 0,$$

$$\int_0^{kl} M'xdx + \int_{kl}^{l} M'xdx + \int_0^{l} M''xdx = 0.$$

Here the values of M' on the left and right of the load are

$$M' = V_1x, \qquad M' = V_1x - P(x - kl),$$

while the value of M'' is $-H(y - y_1)$ for all sections. Introducing these and replacing y by its value in terms of x, the integration gives three equations whose solution furnishes the values of V_1, H, and y_1. Then from (1) and (2) of Art. 94, the values of V_2 and y_2 are deduced. The results are as follows:

$$y_1 = \frac{10k-4}{15k}h, \qquad y_2 = \frac{6-10k}{15(1-k)}h, \qquad (1)$$

$$V_1 = P(1+2k)(1-k)^2, \qquad V_2 = P(3-2k)k^2, \qquad (2)$$

$$H = \frac{15}{4} \cdot \frac{Pl}{h}(1-k)^2k^2, \qquad (3)$$

from which numerical values may be computed for any given k.

When y_1 and y_2 have been computed R_1 and R_2 may be found graphically, as also V_1, V_2, and H, provided that the point d, where the lines of action of R_1 and R_2 intersect that of P is known. Here, as in Art. 77, the locus of d is called the reaction locus and it is fully determined by the ordinate q in Fig. 132. From similar triangles,

$$\frac{q-y_1}{kl} = \frac{V_1}{H}, \qquad \text{whence} \quad q = \frac{6}{5}h, \qquad (4)$$

and hence the reaction locus is a horizontal straight line drawn at a height $0.2\,h$ above the crown of the arch.

The ordinates q, y_1, and y_2 completely determine the rectilinear sides of the special equilibrium polygon for a single concentrated load P. The line a_1d represents the line of action of the left reaction R_1 whose magnitude is given in the

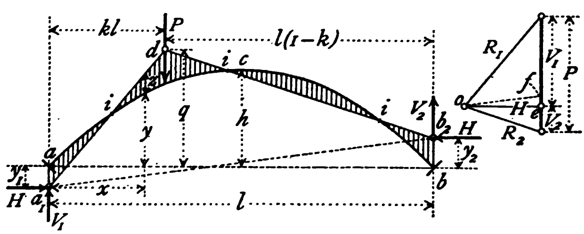

Fig. 133.

force polygon at the right in Fig. 133. If r_1 be the length of the perpendicular from the center of the left support to the line a_1d, the moment at that support is $M_1 = R_1r_1$. If R_1 be resolved into its components V_1 and H, the lever arm of V_1 is zero and that of H is y_1, hence $M_1 = Hy_1$. Similarly, the moment at the right support is $M_2 = Hy_2$. In Fig. 133, k equals 0.3 and from equations (1), $y_1 = -0.222h$ and $y_2 = +0.286h$. The moments M_1 and M_2 have the same signs as the corresponding ordinates since H is positive. The formulas (1) show that M_1 is zero when k equals 0.4 and that M_2 is zero when k equals 0.6. The closing line of the moment diagram is the axis of the arch and if the ordinates z lie above the parabola the bending moment $M = Hz$ is positive, and when they lie below it M is negative. There are three points of inflection i where the moment changes sign on passing through zero. Fig. 132 shows that it is possible to have four points of inflection for a single load.

The reactions V_1 for any number of loads have the same line of action as that of their resultant whose magnitude equals the sum of the individual reactions V_1. The same statement applies to V_2. Because the reactions H have different points of application the line of action of their resultant is found by multiplying each H by the corresponding y_1 or y_2 and dividing the sum of the products by the magnitude of the resultant.

Equations (2) show that the values of V_1 and V_2 are not equal to the corresponding reactions for a simple beam of the same span as was the case with the three-hinged and the two-hinged arches. The difference is given in Fig. 133 by ef on the load line, the ray of being drawn parallel to the line a_1b_2. Its value is $P(k - 3k^2 + 2k^3) = Pk(1 - k)(1 - 2k)$, or, expressed in terms of M_1 and M_2 at the supports, it is

$$(M_2 - M_1)/l = -H(y_1 - y_2)/l.$$

It may aid the student to imagine that the arch has a rigid projection or arm extending from a to a_1 at the extremity of which the reaction is applied. A part of the abutment or pier acts as such an arm. The line bb_2 may be regarded as a similar arm.

Prob. 95. Compute the horizontal and vertical components of the reactions for a panel load of 1 kip at each panel point of an arch rib with fixed ends which has the same dimensions as the two-hinged arch in Art. 77. Compare the corresponding reactions of the two arches graphically.

ART. 96. POSITION OF THE LIVE LOAD.

Equation (4) in the preceding article shows that the locus of the point of intersection *d* of the reactions and the load in Fig. 133 is a straight line parallel to *ab*. The reaction locus thus obtained is, however, not sufficient to determine the directions of the reactions for any load. The envelope of the lines of action of the reactions consists of two hyperbolas which are tangent to each other at a point distant one-third of the rise

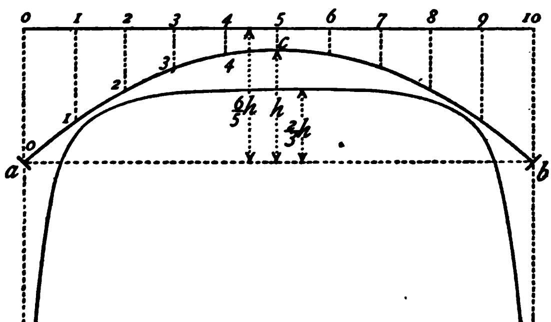

Fig. 134.

below the crown. The verticals through the supports are asymptotes to the curve. The horizontal reaction locus and the envelope curve are both shown in Fig. 134. The curve may be most readily constructed by drawing a number of its tangents with the aid of the following table.

k	y_1/h		k	y_1/h	
0	$-\infty$	1.00	0.40	0	0.60
0.05	-4.667	0.95	0.50	$+0.133$	0.50
0.10	-2.000	0.90	0.60	$+0.222$	0.40
0.15	-1.111	0.85	0.70	$+0.286$	0.30
0.20	-0.667	0.80	0.80	$+0.333$	0.20
0.25	-0.400	0.75	0.90	$+0.370$	0.10
0.30	-0.222	0.70	1.00	$+0.400$	0
0.35	-0.095	0.65			
	y_2/h	k		y_2/h	k

To find the point of division of the loads which produce the greatest positive and negative bending moments at a given section of the arch rib, a line is drawn through the center of moments tangent to the curve; its intersection with the reaction locus gives the required point. When two such tangents can be drawn there are two points of division. As the envelope curve takes the place of the end hinges the method of determining the position of the live load causing the greatest moments and shears is in all other respects exactly like that for two-hinged arch ribs, and therefore no additional example is needed.

Prob. 96. A parabolic arch rib has a span of 180 feet and a rise of 30 feet. The floor system is divided into 12 equal panels. Find the live loading which produces the greatest positive and negative moments and shears in sections 1 to 6.

ART. 97. DETERMINATION OF STRESSES.

After the position of the load is found which causes the greatest positive or negative moment, the moment itself is most conveniently determined by computation. A table of reactions should be prepared similar to the one in Art. 77 with additional columns for y_1, $Hy_1 = M_1$, y_2 and $Hy_2 = M_2$. The bending moment at any section is

$$M = M_1 + V_1x - Hy - \Sigma P(x - kl).$$

After V_1, V_2 and H are computed for a load of unity at each panel point, the thrust is found by means of a diagram and table like those given in Art. 79, and the shear is found at the different sections in the manner explained in Art. 83.

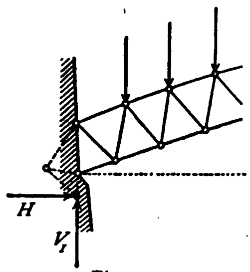

Fig. 135.

When the arch rib has open web bracing the dead load stresses may be obtained graphically by a stress diagram, it being necessary however to introduce an auxiliary truss composed of one or two triangles to connect the end of the trussed arch with the point of application of the horizontal reaction as illustrated in Fig. 135.

Prob. 97. Compute the greatest positive and negative moments and shears in sections 5 and 10 of the arch in Prob. 95.

ART. 98. RIB SHORTENING.

The effect of the thrust along the axis of the arch is similar to that for the two-hinged arch (see Art. 80) but since the ends are fixed the horizontal reactions are applied some distance above the line ab in Fig. 136. Rigid arms may be imagined as

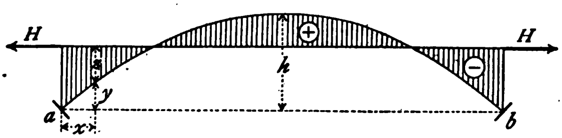

Fig. 136.

connecting the rib with the points of application of H. Since there are no other external forces their lines of action must coincide and consequently $y_1 = y_2$. On applying this condition to equation (3) of Art. 94, it becomes,

$$\int_0^l (y - y_1)\,dx = 0, \quad \text{or} \quad y_1 = y_2 = \tfrac{2}{3}h, \tag{1}$$

and hence the moment M'' is $Hz = -H(y - \tfrac{2}{3}h)$.

To find the value of H due to the rib shortening it may be noted that the last term in equation (4) of Art. 94 gives the horizontal displacement due to H, and that, as in Art. 80, sl/E is the equal displacement due to the thrust along the axis. Hence

$$-H\int_0^l \frac{(y - \tfrac{2}{3}h)y\,dx}{EI_c} = \frac{sl}{E}$$

the integration of which gives

$$H = -\frac{45\,sI_c}{4\,h^2}, \tag{2}$$

which is the horizontal thrust due to rib shortening.

This formula is applied in the same manner as that illustrated by the example in Art. 80 for the two-hinged arch. On comparison it is seen that the value of H for an arch with fixed ends is six times as great as for one with two hinges. Further, the largest ordinate z for positive moments is one-third as great, and hence the greatest positive moment is twice as great in arches with fixed ends as in those with hinged ends. At the fixed ends where the maximum moments are negative the bending moment due to rib shortening is four times as great as that at the crown for two-hinged arches.

Since H is negative the moments have the opposite signs from the ordinates z, but as there is usually no difficulty in determining the signs of these moments such as might occur in dealing with the live load, no special inconvenience or liability to error will result from this fact. The signs of the moments are marked on Fig. 136.

ART. 99. TEMPERATURE STRESSES.

Since changes in temperature develope equal and opposite reactions H their lines of action must coincide and be above the axis ab in Fig. 136 at a distance of two-thirds of the rise of the arch for the same reasons as those given in the preceding article. For a fall of t degrees in temperature the axis would be shortened by an amount equal to ϵtl (see Art. 81) provided one end were free to move. Substituting the values of $M'' = -H(y - \frac{2}{3}h)$, $ds = dx \sec i$, and $I = I_c \sec i$, as before, the last term in equation (4) of Art. 94 is equal to ϵtl, or

$$-H\int_0^l (y - \tfrac{2}{3}h) y dx = EI_c \epsilon tl,$$

and after integrating and reducing, there is found for a fall in temperature,

$$H = -\frac{45\, EI_c \epsilon t}{4h^2}, \qquad (1)$$

and similarly, for a rise in temperature,

$$H = +\frac{45\, EI_c \epsilon t}{4h^2}. \qquad (2)$$

For falling temperature the bending moments have the opposite signs from those of the ordinates z, as illustrated in Fig. 136, while for rising temperature they have the same sign.

The above values of H bear the same relation to its values for an arch rib with two hinges as was found in the preceding article to exist between the corresponding thrusts H due to the shortening of the rib axis; that is, a given change in temperature causes six times as great a horizontal reaction in an arch rib with fixed ends as in one with hinged ends. The moment at the crown is twice as great while that at the fixed ends is four times as great as at the crown of the rib with two hinges.

The stresses due to temperature in an arch rib with either solid or open webbing may be found by the methods already given for the two-hinged arch.

Prob. 98. The central arch rib of the St. Louis bridge has a span of 519.233 feet and a rise of 47.31 feet. The chords at the center have a section area of 67 square inches and are 12 feet apart. The coefficient of expansion for a change of 80 degrees Fahrenheit was taken at 0.000527 and the coefficient of elasticity of the steel tubes composing the chords was 27 000 000 pounds per square inch. The thrust at the piers was found to be 204.9 tons and the moment −6747 tons-feet. What would be the values of the thrust and moment if the arch were parabolic with the same rise and with its moment of inertia varying as that assumed in deducing the preceding formula?

ART. 100. CONCLUDING REMARKS.

The theory of the three-hinged arch, presented in Chap. V, is in all respects more exact and satisfactory, than that of the arch with two hinges or the arch with fixed ends. This is the case because the introduction of the hinge at the crown renders the reactions statically determinate. Thus, whatever be the loading, the reactions due to those loads are found without any assumptions derived from the theory of elasticity, and the resulting stresses are as closely exact as in the case of a simple beam or truss.

It is generally supposed that a three-hinged arch is not subject to stresses due to changes in temperature. In strictness, however, such stresses will occur, for a fall in temperature causes a decrease in the rise of the crown, and, as the span does not change, the horizontal thrust will be increased. Likewise a rise in temperature will decrease the horizontal thrust. These changes in the reactions will modify the existing stresses to a slight extent particularly in arch ribs of shallow depth.

For the two-hinged and fixed-ended arch ribs marked stresses due to changes in temperature occur, those for the latter being at certain places from two to four times as great as those for the former, and both far exceeding those that occur in the three-hinged structure. In respect to temperature stresses, then, the three-hinged arch takes the first rank while the arch with fixed ends has the least advantage.

In regard to stiffness the reverse is the case, for the restraint of fixed ends lessens the deformation that would otherwise occur. Both under live load and under changes of temperature the arch with fixed ends is subject to less deflection than the two-hinged arch, while the latter is also materially stiffer than the arch with three hinges. Hence it is that while the three-hinged structure may be suitable for a highway bridge of light traffic, it may fail to give satisfaction if used for a railroad bridge of long span under heavy traffic.

Both the two-hinged and fixed-ended arches are statically indeterminate structures, that is, the reactions and stresses can only be determined by taking into account the deformation of the material, and this is always supposed to occur within the limit of elasticity. Hence the common theory of the arch rib is subject to all the imperfections of the theory of continuous structures. Many of the objections against continuous bridges, stated in Art. 12, apply with equal force to these two forms of arches; in particular the erection demands the most careful workmanship, and yielding supports will cause great changes in stresses. Further, if loads should ever be applied which cause the stresses to exceed the elastic limit of the material, the entire theory fails and it is impossible to predict the degree of security of the structure.

Undoubtedly many more arched bridges will be built in the future than in the past, but in view of the arguments here set forth it is thought that the main development should be along the line of the three-hinged structure. The study of the best proportions, and of the most advantageous method of combining arch ribs and spandrel bracing is yet in its infancy, but through this it may be possible to render the three-hinged arch more advantageous in regard to stiffness and to readily apply to cases where the two-hinged form is now used. Only in instances where abutments of solid rock are at hand and where the traffic is very heavy can the use of the statically indeterminate forms be regarded as entirely legitimate in theory and satisfactory in practice.

INDEX.

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